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ABSTRACT

This document consists of seven chapters. The first chapter analyzes the modeling process and gives reasons for introducing a modeling approach in classrooms, arguing that students will develop more confident and positive attitudes toward mathematics. The next five chapters contain case studies, which focus, respectively on: whether it is more economical to rent or buy a television set; how pricing policy affects profit; how a rear window wiper should be positioned on an automobile to sweep the maximum area; the mathematics of the long jump; and which of three schools should be shut due to declining enrollments (by calculating traveling time by different modes of transportation). Each of these chapters includes: an introduction to the problem to be modeled, with prerequisites and the target age-range (and learning objectives, when applicable); list of assumptions on which the model is based; an influence diagram showing the relationship between factors; model development (which may consist of several parts); and a concluding section which draws out implications and gives ideas for further work. Assessment items are included for several models. The final chapter contains a list of models which could be used in the classroom together with a list of references and a bibliography. (JN)

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1. MODELLING FOR SCHOOLS/COLLEGES

edited by

Ramesh Kapadia

Huw Kyffin

South Bank Research Series in Mathematical/Computing Education

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Department of Mathematical Sciences & Computing

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Ramesh Kapadia.
Huw Kyffin.

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INTRODUCTION

There has been much discussion in the past decade about the place of modelling in mathematical education, culminating in the International Conference on Mathematical Modelling at Exeter University in 1983. A journal has been initiated in this area, "Teaching Mathematics and Its Applications", following from the "Journal of Modelling for Teachers". After many years of heart-searching debate about the place of applied mathematics, it seems that modelling can provide the thrust of new ideas required to revitalize this area of mathematics in the curriculum.

However, much of the detailed work in this area has been at the undergraduate level, when students have a good range of techniques at their disposal. If calculus, exponentials, algebra or statistics are required it is likely that the students will have met the ideas before. Hence the principal focus has been on the modelling activity itself, there has been little discussion about the role modelling can play in helping students learn mathematics and acquire confidence in applying it.

The work which has been done in this area has been united under the banners of "mathematics across the curriculum" or "problem-solving". While much useful work has been done in these areas to help children appreciate the development of mathematics in varied situations the framework has been rather haphazard. A more coherent framework is emerging via the field of mathematical modelling.

The models presented in this book go through the modelling process in detail to throw light on the problem under investigation. They have been developed by practising schoolteachers to be used with their own classes. Each stage in the modelling process is considered carefully, from the initial formulation, assumptions and influence diagram, to the development of the model and its validation, often including sensitivity analysis. Thus the models are readily usable in the classroom situation with secondary school pupils in the age range 11-18.

The opening chapter begins by analysing the modelling process, utilizing schemes developed in the current literature. It goes on to give the reasons for introducing a modelling approach in the classroom. It is argued that children will develop a more confident and positive attitude to mathematics.

Five models are presented. For convenience a brief synopsis of each of the models is given here. The same basic format is followed:

1. Introduction.
2. Assumptions.
3. Influence Diagram.
4. Model Development, perhaps in several parts.
5. Conclusion.

Bell considers whether it is more economical to rent or buy a television set. The basic model assumes that cash is available, a second model assumes that money is borrowed to be repaid on simple interest, while the final model considers compound interest. It is suitable for pupils in the age range 14-18. Pupils gain practice in formulating simple linear equations and using line graphs to interpret the results. Interesting questions arise about the validity of the derived equations. The model also shows the break-even point between buying and renting, providing numerical answers to the questions posed; more significantly the role of the assumptions in the answers obtained is highlighted. Sensitivity analysis shows the effect of varying assumptions like the time period of a loan or a different interest rate.

Messer looks at how pricing policy affects profit. This leads to a quadratic equation, from which the optimum selling price and profit can be obtained. It involves construction of equations, algebraic manipulation, and maximization. The results are interpreted and the sensitivity of the main variable is discussed.

Sheath investigates how a rear windscreen wiper should be positioned to sweep the maximum area. In the initial development it is assumed that the wiper blade runs along the arm, but the later model relaxes this requirement. Simple trigonometry yields an equation for the area swept but it involves several variables which make maximization difficult. A numerical and graphical approach is adopted : with the growing availability of calculators and microcomputers this offers a valuable opportunity for this approach. Measurements from photographs of car rear windscreens are included to allow practical validation. The numerical approach also facilitates sensitivity analysis.

Sheridan models one of his own schoolboy pursuits, long-jumping. He begins by considering how the height reached affects the distance jumped; clearly speed also helps and this is considered in the second model; the final model takes account of the fact that the centre of gravity is lower at the end of the jump than at the beginning. This interesting development brings out the influence of the assumptions on the derived answers. It provides practice in algebraic manipulation, graphical interpretation and solving differential equations; as such it is suitable for older pupils.

Close considers the problem of which of three schools should shut in the light of falling rolls. The mathematical ideas used are kept simple to make the work accessible to younger pupils, in the age-range 11-16. The model relies on simple calculation and reading and interpreting graphs. The basic idea is to calculate travelling time by different modes of transport. Assumptions are made about walking speed, waiting time and variability of bus journey time. Then sensitivity analysis is done on the final equations to show the effect of varying these assumptions.

Each of the models was developed by a practising teacher whilst studying for an M.Sc in Mathematical Education at the Polytechnic of the South Bank. They have been edited to standardize the presentation whilst preserving the essence of the teacher's work.

The final chapter contains a list of models which could be used in the classroom, together with a list of references and a bibliography for further investigation.

1. THE MODELLING PROCESS

If we consider the uses to which mathematics is put in the real world, we can see that the mathematical formulation of a problem, whether it is an equation, a graph or a matrix is a concise summary of some aspect of the real world. As Cockcroft (1982 p. 1) says 'the perceptions of the usefulness of mathematics arise from the fact that mathematics provides a means of communication which is powerful, concise and unambiguous'.

Mathematics is widely applicable because it provides the language in which a situation can be understood and it provides appropriate ideas to help solve problems. A question, which is perhaps rather vague, can be formulated in clear and meaningful terms; from this mathematical framework, definite results can be obtained. Often the same type of formulation is applicable in a wide variety of contexts. For example, exponential functions are used in the study of radioactive decay, biological populations, heat loss and drug concentration. After definite results are obtained these can be compared with the real situation to provide a validation of the formulation, or perhaps suggest a different formulation.

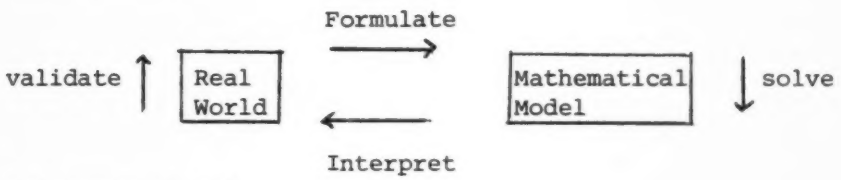
The production of a concise and unambiguous mathematical description of an aspect of reality is the basic modelling problem. A real problem is often very difficult to formulate unambiguously. Many interesting problems may not seem to be capable of concise formulation, while those which are may not be interesting. Even when a problem has been formulated the mathematical techniques necessary to solve it may not be available. This arises when a student comes up against a problem which requires algebra for its solution, before it has been studied. But also the need for new techniques is a source of progress in mathematics - as Newton's invention of calculus to deal with continuous motion demonstrates. Mathematical modelling is thus a necessary part of mathematical research.

It is even more a necessary part of research in physics, chemistry and the other sciences. Hence it is vital to stress the role of modelling in teaching these subjects. It is an instructive exercise to teach the kinetic theory of gases from a modelling standpoint. Students normally tend to parrot the assumptions of this theory which they are allowed to learn without thinking at all why anybody should assume such a bizarre collection of hypotheses. If the notion of modelling is introduced, then the hypotheses of the kinetic theory are seen in their proper light, as attempts to provide a microscopic model for macroscopic behaviour. The fruitfulness of using such models is illustrated time and time again in the history of science, particularly in the way that it is possible to maintain two different and even contradictory models of the same thing for explanations in different fields. The atom may be regarded as an incompressible billiard ball (for kinetic theory) and as a little solar system (for chemistry). An atom cannot <u>really</u> be both of these things at once, but if teachers adopt a modelling approach then it is not an unreasonable strategy to emphasise one aspect of the properties of an atom in one context and another aspect in a different context.

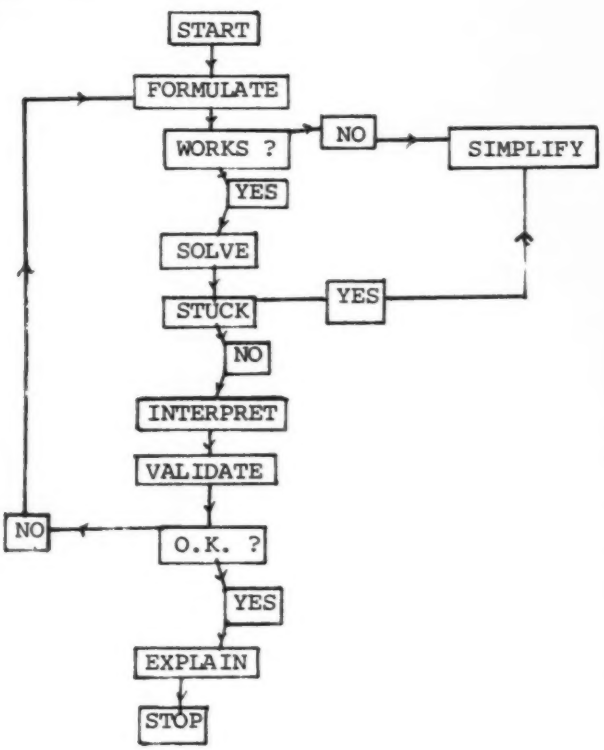
Similarly, the biologist managing forests or fish farms, the civil engineer estimating the safety of structures, the economic planner advocating a fiscal policy and the marketing manager planning a sales drive, all use models of the way their systems behave. The different aspects of the modelling process will vary in their relative importance according to the situation being modelled. In areas like physics and engineering the formulation of the model may be relatively straightforward, but the demand is for high accuracy and precision in the predictions given by the model. For disciplines where the causal links are more complex the formulation may be extremely difficult or at least highly speculative. Here the modeller must be very explicit about the assumptions involved, since it is with poor assumptions that a model may lead to poor predictions.

We have mentioned two stages of the modelling process: the formulation of a model, and the comparison of its prediction with the real world - validation. Various writers have analysed the complete modelling process, and produced models of modelling. The following schemes are examples of some in the current literature. One of the simplest descriptions is by Burghes; it encapsulates the main features in a diagram which is easy to remember. Burkhardt has also developed a flow chart of the modelling process.

1. Burghes (1980)

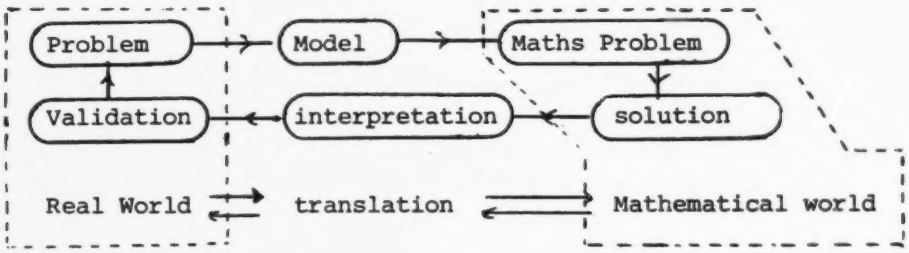


2. Burkhardt (1979)

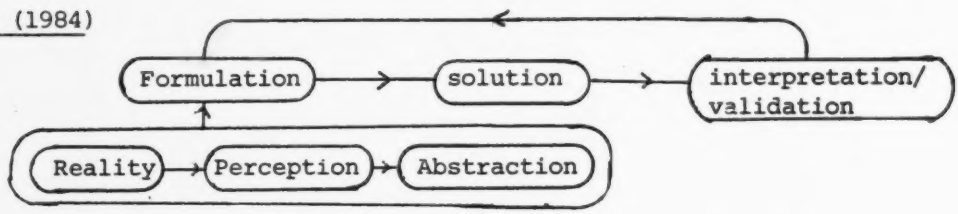


In actual practice modelling a situation is not really such a linear or clear-cut process, yes/no answers are often inappropriate. A few other attempts to capture the modelling process are given below. Perhaps it is a matter of taste which one chooses.

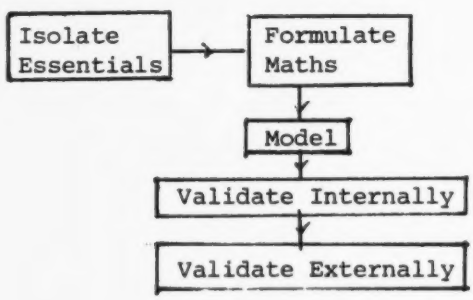
3. Penrose (1978)



4. Oke (1984)



5. Ford & Hall (1970)



Each of the schemes provides a simple overall portrayal of the modelling process. However, modelling is a creative process, as is painting, and the links between any two activities are highly complex often involving the carrying out of several operations simultaneously; relationships created at different points of the model's construction are often used together in one jump in reaching new decisions. Relationships are often written in various forms as the solution develops, and the need for further understanding of a problem often only becomes apparent as some mathematics is used. Consequently, the formulation and solution stages are intimately interwoven.

The modelling approach stresses formulation and validation. Formulation, because much the most challenging and interesting part of the problem is finding out how an initially vague question can be made precise enough to be cast in a mathematical form, how to find a measure for what we want, how to establish criteria for solution, and so on. Attempts at formulation can take a long time and journeys may be made along many false pathways before the route to a solution is found. Important devices to help the formulation process consist of listing explicitly the assumptions used and drawing influences diagrams to connect the variables involved and to help in finding the relationships between them.

Oke has devised a more extensive list, which aims to guide in the initial formulation - solution stages:

Establish a clear statement of objectives

State obvious or natural objectives - but try and be fairly precise.

Don't write a vast list of features

A large list of everything you can think of only serves to confuse. However, do write down any features considered important - then decide which to consider in detail.

Simplify

Build up very gradually. Make guesses, make assumptions, add restrictions. Lump components (attributes) together and treat as single component or if original highly complex, break-down into simpler problems and treat each separately.

Get started with maths as soon as possible

Identify a few variables, parameters, constants. Write down one or two obvious mathematical relationships.

Keep mathematics as simple as possible.

Carry out some mathematics on initial relationships

This itself generates more variables, constants and relationships.

Got a solution yet ?

If relationships so far do not satisfy objectives then create more obvious mathematical relationships and combine (mathematically) with those you already have.

Know when to stop

Do not seek perfection. A mathematical model is only an approximation to reality. There are no right or wrong answers, only good or bad.

Interpret your solution

Use common-sense.
Test for self-consistency (i.e. no mathematical or logical mistakes, insert a few reasonable looking values).

Validate your solution

Use known data to test model's ability to predict or verify over a wide range.

If stuck

Observe practical situation if possible or carry out a 'thought experiment' and imagine what is happening. Plot measured values, form empirical relationships.

Have frequent rests

Do not spend more than about one hour at a time when starting a new problem.

As modelling is a looping process there is not necessarily any reason to stop modelling. It is this feature of modelling which makes it such a useful method. We can increase the sophistication of our model up to the limits of the techniques available or try to develop new techniques. The decision when to stop the model may be fairly arbitrary, and may be because of fatigue, boredom or diminishing returns. One might like to model criteria for when to stop modelling, for example if the gain from the next model is less than the effort required to produce it, then stop at that model. Clearly there are important pedagogical implications in that students will gain a better understanding from engaging in the modelling process.

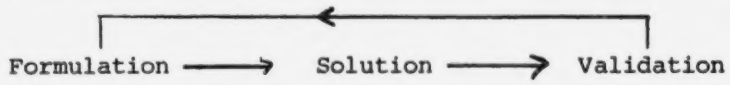
2. MODELLING IN THE CLASSROOM

Teachers may ask why it is a good idea to teach modelling in school. At some stage in the training of mathematicians and users of mathematics it becomes necessary and is usually taught at undergraduate level. We would argue that it is desirable to teach through modelling much earlier.

The emphasis at school must be on teaching mathematics through modelling rather than on teaching modelling. The latter is important at degree level where students are being trained to use mathematics professionally with careers possibly involving modelling. However, most students at elementary levels will not become professional mathematicicians so the main argument must be that modelling is a useful strategy for teaching mathematics.

This is not to underestimate the importance of learning important general strategies in how to model a situation, for many of these strategies apply to situations where the problem is not only mathematical. As far as the mathematics teacher is concerned, however, the main benefit of adopting a modelling approach would be that students would become more confident and able to see the mathematics involved in any problem more clearly. It provides an opportunity to realize a fundamental aim in mathematical education, using mathematics in real situations.

We have given above various models of the modelling process. Oke's is perhaps the simplest:



Traditional mathematics courses concentrate almost entirely on the solution. A problem is presented already formulated, the student's task is to solve it, and very little attempt is made to match up the solution with the real world. In so far as such attempts are made, the models are so standard that results which deviate considerably from expected values (such as cars weighing 0.5 mg, say) are always the result of an error in calculation rather than a defect in the model.

In the sixth form (16-19) pure mathematics is highly technique-oriented and applied mathematics, despite the recent rise of statistics, remains rooted in the application of standard techniques to mechanics problems. It may be argued that the constraints of examination syllabi mean that not enough time is available for what is necessarily a more discursive approach and for younger pupils that the techniques involved in modelling may be beyond their mathematical capability. We believe that for advanced students the time set aside for modelling will be more than rewarded as they will develop far greater confidence in understanding and applying mathematical techniques to a variety of situations. The use of modelling will improve the student's relational understanding. For the younger pupil a little imagination will generate situations which are interesting and tractable.

The articles by Nelson (1977) and Pollak (1969) give excellent examples that could be used, as do the essays in this book. The journal "Teaching Mathematics and its Applications" and its predecessor the "Journal of Modelling for Teachers" also provides examples of elementary models. The Spode group (1983) have recently published a set of models which only use CSE mathematics, to complement their earlier publications utilizing 'O' level mathematics.

There are important reasons for using a modelling approach. It demonstrates the usefulness of mathematics in real situations and encourages mathematical thinking. Moreover the student will understand how certain kinds of mathematical structures naturally arise, and be able to see mathematical concepts in context. It shows clearly how the selection of appropriate, simplifying assumptions aids solutions of a problem. In mechanics, for example the "frictionless plane" is a useful modelling device not some kind of artefact.

The models in this book argue collectively for approaches to mathematics through modelling to be adopted at all levels of teaching, and we feel that there is no necessary lower limit at which this approach can be usefully undertaken. Much of the activity of the modeller resembles that of the problem solver, particularly in that both have to discover the method of solution before solving the problem rather than merely apply a known algorithm.

The chief difference between problem-solving and modelling can be found in the types of problems being tackled. For the modeller the problem will not arise in mathematics itself. In problem-solving the problem is often mathematically well-defined, though perhaps not so for the student. In modelling, while the problem may be clear, the relevant mathematics is initially rather ill-defined and one seeks to make sensible simplifications and gain a better understanding of the problem bfore trying to apply mathematical techniques.

Teachers will wonder how these ideas may be incorporated into existing syllabi. We believe that this can be done but is not easy. The strongest impetus for the widespread adoption of modelling approaches would be examinations which include modelling questions. Meanwhile modelling is an alternative method of introducing a topic or a method of consolidating and uniting an area of work; more vitally it shows the applicability of mathematics.

With this in mind, an important consideration that arises is the mathematics which models could help to illustrate and consolidate. Graphical work often arises in a modelling situation. A graph is a powerful medium of communication and it provides a means of testing a hypothesis when variables have been transformed to give linear relationships; one can also investigate small changes in linear relationships. Geometrical and trigonometric ideas can be utilized in spatial problems. An optimization problem can often be solved by using calculus. Probabilistic ideas are applied in simulation, while the gathering of statistical data often illuminates important features of a problem. It is clear that a wide variety of mathematical ideas do arise naturally in modelling.

Possibly the biggest implication for mathematics pedagogy is that the modelling lesson has to include discussion. This is one of six areas that Cockroft (1982) stresses as being vital components of good mathematics teaching practice. Mathematics teaching at all levels should include opportunities for : (paragraph 243)

> exposition by the teacher;
> discussion between teacher and pupils and between pupils themselves;
> appropriate practical work;
> consolidation and practice of fundamental skills and routines;
> problem-solving, including the application of mathematics to everyday situations;
> investigational work.

Indeed each of these components will naturally arise if a modelling approach is adopted.

Real discussion can take place in the formulation process because there are no right or wrong answers and students' suggestions have as much validity as the teacher's. The only way in which the teacher has expertise at this stage is that, as an experienced mathematician, the teacher should have an awareness of promising lines of attack, and should be able to identify approaches which will not demand mathematical sophistication beyond that possessed by the students. Apart from this gentle hand of the teacher's on the rudder, the formulation process should be an open discussion out of which the main issues of the problem emerge - the idealisations and simplifications of the initial model should follow. Formulation sessions can be either in small groups or as a whole class. The experience of such sessions is that tangible progress towards the solution of the problem may be slow in coming but comprehension of the problem develops enormously. What happens during the formulation phase is that every idea that is thought of is challenged, examined, dissected and the concept embodied within it clarified; next the relationships between concepts are analysed. This process goes a long way to counter the problem noted by the National Secondary Survey that "the potential of mathematics for developing precision and sensitivity in the use of language was underused" and it also develops ability "to say what you mean and mean what you say" (Cockcroft 1982 p.72).

It is not necessary to have an explicit modelling approach to mathematics teaching in order to involve the students in practical work, but the approach of modelling with its emphasis on setting up and testing hypotheses, and identifying important variables lends itself naturally to practical work. Information gathering or drawing or building a model is often necessary before a problem can be fully formulated. It can also be required for validation.

The modelling process itself is open-ended and investigatory in nature, but some problems involve a great deal of exploratory work. This is a third area in which Cockcroft recommends that mathematics teachers should provide opportunities. Modelling real situations will naturally give students the opportunity to decide what data to collect and how to go about it. In so doing they will be able to construct hypotheses and test them. In deciding whether to rent or buy a television for example, the students need to get their own data from local shops.

We have already mentioned the relation between modelling and problem-solving, and Cockcroft mentions the difficulty students have because 'the solution of a mathematical problem cannot begin until the problem has been translated into the appropriate mathematical terms' (page 73). This is precisely the problem of formulation in modelling.

Modelling can also help the teacher, to provide practice in skills and routines (page 73). It is surely better to practice these in context rather than in isolation. Any model will be rich in possibilities to practice techniques amongst the following skills which are likely to arise: collection and tabulation of data, calculation, drawing graphs, algebraic manipulation, statistical calculation. Finally, modelling also provides the opportunity for a teacher to explain a technique or concept, where appropriate and certainly in context. This is the sixth area listed by Cockcroft, namely exposition.

Thus in the modes of teaching highlighted by Cockcroft as needing particular attention, modelling is a technique which gives much scope to the teacher particularly with regard to the need for discussion, the role of problem-solving and investigative work; it has more to offer than traditional teaching methods. Crucially modelling shows the power and applicability of mathematics, it provides a link between mathematics and experience.

3. THE MODELS

The models presented in this book were written by practising mathematics teachers at schools and colleges as part of an M.Sc course in Mathematics Education at the Polytechnic of the South Bank. A mathematical modelling course was one of the four compulsory courses and teachers had to produce their own models and to indicate how they would use them in the classroom. The emphasis was therefore on using mathematical techniques appropriate for their students. Many of the teachers on the course felt that mathematical modelling provided the most challenging and new part of the work and that it was the subject in which their own preconceptions of the nature of mathematics teaching were most challenged and modified. For many of the teachers it was the first time for years that they were confronted with problems which were genuine i.e. for which the method of solution was not apparent and had to be found first. Also the wide range of applications of the models had a salutary effect on those brought up to regard applied mathematics as merely a branch of mechanics. Moreover the transition from a relatively vague problem to an equation which could be tested gave a tremendous sense of achievement, even if the problem was fairly trivial and the solution of it not very important.

Examples of situations the teachers considered on the course included: the best strategy to sweep up autumn leaves, the best angle for a turntable arm on a record player, the action of a kidney dialyser, when to turn right at a T-junction, the optimum amount of drink to order for a student party, heating a baby's milk. The reader may like to attempt to formulate and solve them.

The models represent a range of possible applications of modelling, both in terms of the mathematical techniques used, and in terms of the areas from which the problems arise. Two of the models (Bell and Close) use quite elementary mathematics accessible to most pupils, while the others can be used with an above average class for much of their development. Indeed it is possible to use parts of a model with pupils and return later when they have studied further mathematical techniques. The problems have been specifically selected by teachers in areas which students have related to and found interesting.

Messer approaches costing problems, in particular - how does a shop price its goods, a question which affects everybody. Sheath deals with the design of car windscreen wipers. On the other hand Bell's model demonstrates that the decision many people take (to rent a TV set) is probably not the financially wisest one to choose. Thus the use of modelling may confirm that a strategy is a wise one, or it may give us the information on which to base a change in our behaviour. Close's model is an example of a general type of location problem relating to mergers of schools, a matter of much current concern. Sheridan's analysis of the long jump is just one of many examples of problems in sport which could be modelled. This model uses basic ideas from physics, but others can be more mathematical, for instance the criteria for a goalkeeper moving off his line to block an opposing forward.

One thing which soon becomes clear from each of these models is that the process of modelling generates new thoughts and new insights into the problem. This may be one of the most useful results from modelling. The stimulus that it gives to the student to think mathematically can alter the student's attitude towards the subject itself. Cockcroft is one of many who have noted the unpopularity of mathematics, and the strategies whereby people avoid having to do mathematics. Many authors (e.g. Tobias, Buxton) have discussed the problem of 'Math Anxiety', and it is clear that mathematics does have an intimidating character for many people. The causes given for this are the abstract, impersonal nature of mathematics, and its apparent rigidity and inflexibility. Mathematics is seen as a subject which gives right or wrong answers, and while we have already mentioned this as a source of the power of mathematics, these answers often arise in ways which are totally mysterious to the student. By adopting a modelling approach students see how abstractions arise out of a real, concrete situation. They see that these abstractions arise so that a situation can be made more tractable, rather than the abstraction being in some way truer than the real situation. (Platonism may be a satisfactory philosophy for many mathematicians but as a teaching strategy it leaves a lot to be desired). A modelling approach on the other hand is blatantly empiricist. A model is a provisional hypothesis which is discarded when a better one comes along. We can always improve on our model, but we will at some stage stop modelling a particular problem, not because we have arrived at the Truth, but for a variety of reasons. Our mathematical techniques cannot take us any further, the effort involved in improving is too great for the improvement expected, or we may simply get bored with the problem.

The particular strength of modelling lies in the way in which a student can be motivated by a problem. The solution to the initial problem can generate new problems. It is rare in traditional mathematics for students to be able to immerse themselves in a problem, but in a modelling situation this is precisely what happens. Clearly empirical work in trying out and reporting on using a modelling approach in the classroom is required. Accounts of such experiences are now being published in the journal "Teaching Mathematics and its Applications".

II. THE DECISION WHETHER TO RENT OR BUY A TV SET

Andrew Bell

1. INTRODUCTION

In a society where rental agreements are common, it is an interesting exercise to compare the benefits and disadvantages of buying or renting. This applies particularly to television sets though now it could also be applied to video-recorders. This is an important area to build up consumer awareness.

In introducing this topic to a class, wider issues could be discussed before trying to apply a mathematical analysis. The price of a television varies considerably between a local store and a discount warehouse. However knowledge requires time and money; transport might pose problems. When a set is rented there is usually a guaranteed fast service in case of a breakdown, with an owned set this may not be so easy. Nowadays, however, sets have become much more reliable. When new models come on the market or there is a technological development, a rented set is easier to exchange. In raising money for a loan, banks may demand a guarantee. These are some of the general issues which could form the background for the modelling exercise.

In this model straight line graphs are used to yield useful information for the decision whether to rent or buy a TV set. By detailed formulation and using real data the student can see the experimental nature of mathematical thinking.

Often there is more than one solution, and changes may make today's solution incorrect tomorrow, for instance, a change in the rate of VAT, or in interest rates. However, the model building exercise shows the student how to identify important factors in the decision process and how these factors influence the decision.

The models in this chapter develop through what Morris describes as enrichment or elaboration, starting from very simple models and leading to ones which reflect reality more accurately. The initial model is for CSE students (of average ability); by bringing in factors such as interest on loans, compound interest, length of loan and expected life time of television, the final models are suitable for more advanced students.

This problem lends itself to individual project work whereby students can collect data from TV shops to use in the validation stage.

Please note that the graphs are sketched for illustration, they would need to be re-drawn accurately for classroom use.

Age-Range : 14-18 years

2. INITIAL ASSUMPTIONS

A1) The person concerned has no television set of any sort which is available for trade-in purposes.

A2) The television set to be bought or rented is a new 22" colour model.

A3) Whichever method of obtaining the television set is chosen, an aerial is needed and that the aerial installation charge is independent and fixed and can be eliminated from our calculations.

A4) Whichever method of obtaining the colour television set is chosen, a licence fee will have to be paid and this can be eliminated from our calculations.

A5) Whichever method of obtaining the colour television set is chosen, the cost of insuring the television set is independent and can be eliminated from our calculations.

Note, that while assumptions A3, A4, A5 are reasonable in that the costs concerned are the same whatever method of payment for the set is adopted, they do represent an extra cost, and may therefore have an important marginal effect on whether to obtain a TV set at all. Or, if the person can only just afford a set, when these factors are taken into consideration this would have a large effect on the decision, particularly if a large initial payment is called for.

All the figures are in pound sterling, so the sign is usually omitted.

3. INFLUENCE DIAGRAM

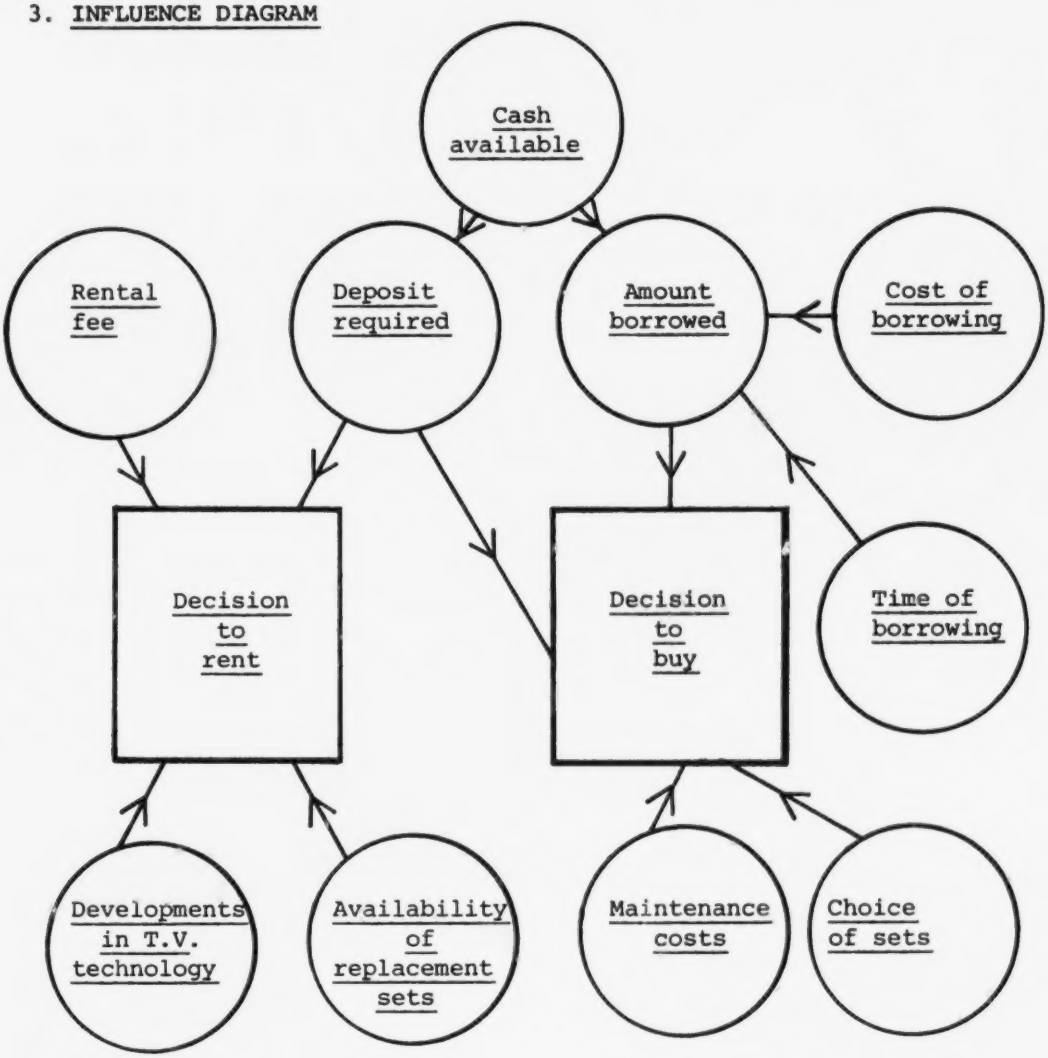


Figure 1 : Influence Diagram

For the different models considered, the assumptions specific to each model simplify this influence diagram.

Thus, in this CSE model the assumption is made that the cash to buy the set outright is available, and, therefore, there is no need to consider borrowing. In the other models, however, the more realistic assumption is made that it is necessary to borrow the money.

From the viewpoint of this modelling exercise, developments in technology, availability of replacements and choice of sets are not taken into account in the formulation. This, of course, is not to deny the possible importance of these factors, but they are not factors which are tractable in a simple model. Readers may be interested in attempting to formulate these factors in further models.

4. MODEL DEVELOPMENT (CSE)

As an elementary start to this model, suitable for C.S.E. students, we will make some simplifying assumptions necessary for a basic understanding of the problem, before attempting to build the mathematical model, which should be a set of logical relationships linking the relevant features of the reality with which we are concerned.

We will assume that there are no financial restrictions at all, i.e. enough money is available to not only pay the rental deposit but also to buy the television set outright if required.

Assumptions specifically for this level.

The rental charge per month remains constant.

The rented set is not changed until it is worn out.

The maintenance charge is independent of the cost of the T.V. set.

The cash necessary for purchase is available in total.

Clearly these assumptions require considerable discussion. Students may be able to provide their own information on how rental costs change or whether the maintenance cost does depend on the cost of the set. Nevertheless they should also see how these assumptions simplify the calculations and give a good starting point.

Rental Costs

Rental charge per month = R pounds
Annual rental charge = 12R
The rental companies usually require a prepayment D equal to 13 or 26 weeks.
The payment during the first year is therefore:

$$D + (12R - D) = 12R$$

Payment during subsequent years is 12R per year, therefore
C_r = total cost for n years $= 12nR$

Cost of purchasing

Let purchase price = P pounds

Let maintenance charge be M per year, although the first year will be free from maintenance charges since it will be under guarantee.

Let C_p = total purchase cost.

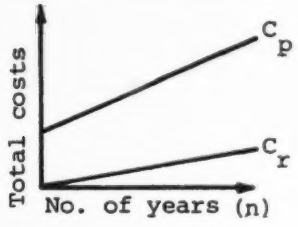
$$C_p = P + (n - 1)M$$

For a given value of n
If $C_r > C_p$ then purchasing is a more economical proposition.

If $C_r < C_p$ then renting is a more economical proposition.

If $C_r = C_p$ then there is nothing to choose between either system.

Comparing Costs



Rental will be more economical for whatever value of n.

Fig 2.

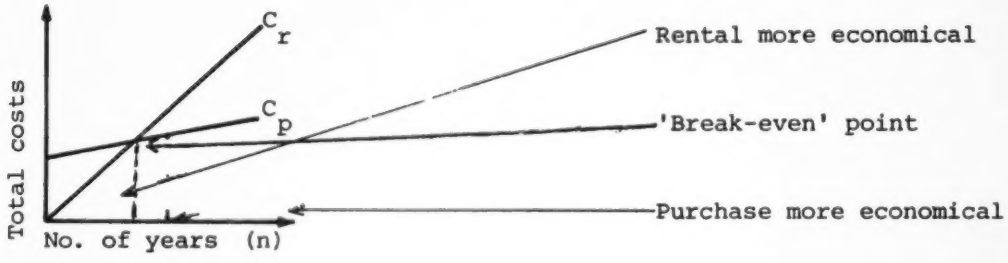


Fig 3 : Comparing rental and purchase costs.

From the equations for C_r and C_p it is clear that the gradient of the C_r graph is equal to 12R, and the gradient of the C_p graph is M.

The first graph would occur if M > 12R, if M < 12R then the second graph would occur. Which is more likely to happen in practice ? This question is answered in the validation below.

Validation.

Information was supplied (in 1982) by Trend Electrical, Dixons, Rumbelows, Currys, Granada.

We can take the following as good estimates of prices:

Average monthly rental charge = £10
Average purchase price = £300

There is little deviation in monthly rental charge: dearest £10.25 at Granada, cheapest £9.95 at Rumbelows.

But there is a large variation in purchase price: cheapest £260 for a Bush at Dixons, dearest £450 for a Sony at Dixons.

The average price for a maintenance agreement is £40 per annum.

Taking the average prices,

Rental cost $C_r = 12nR$

$C_r = 120n$

Purchase cost $C_p = P + (n - 1)M \quad P = 300, \quad M = 40$

$$C_p = 300 + 40(n - 1)$$

$$C_p = 260 + 40n$$

Since the purchase price varies so much it is useful to calculate values for $C_{p.min}$ based on a purchase price of £250, and $C_{p.max}$ based on a purchase price of £450. (Note also that the equation above is not true if n=o).

These lead to the following equations:

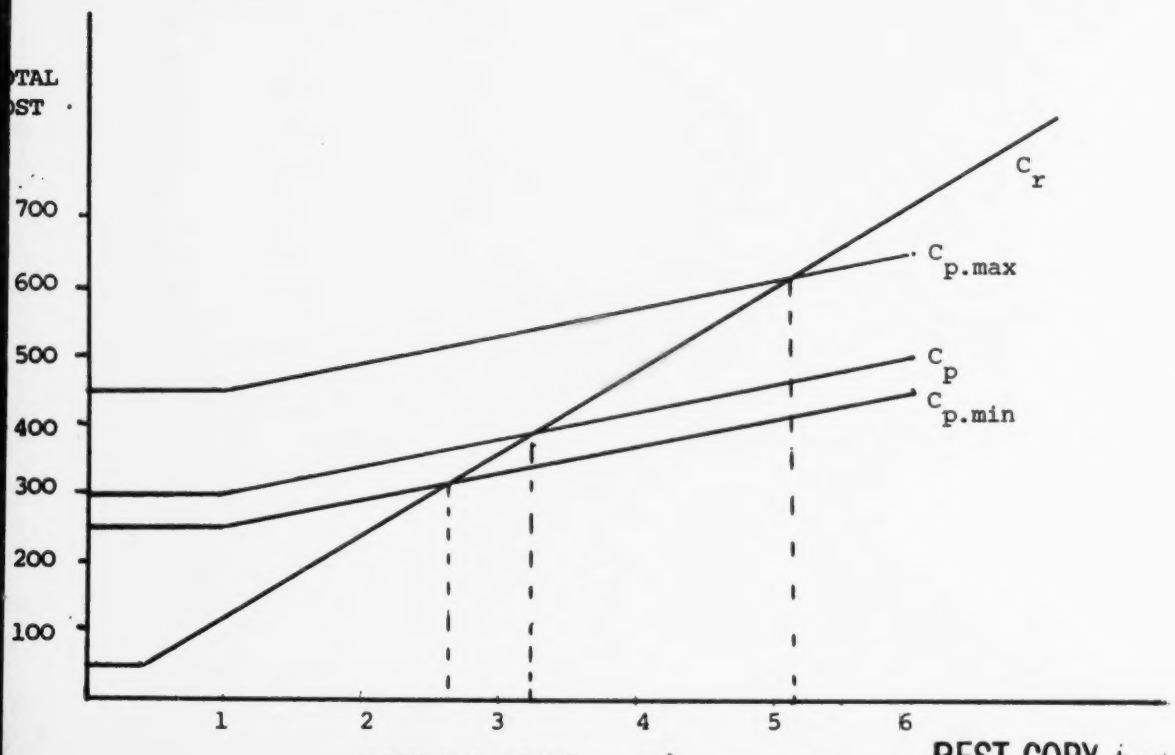
$$C_{p.min} = 210 + 40n$$

$$C_{p.max} = 410 + 40n$$

The results for C_r, $C_{p.min}$ and $C_{p.max}$ are tabulated below and drawn on Fig 5.

n	0	1	2	3	4	5	6
C_r	40	120	240	360	480	600	720
$C_{p.min}$	250	250	290	330	370	410	450
C_p	300	300	340	380	420	460	500
$C_{p.max}$	450	450	490	530	570	610	650

FIGURE 5 CSE MODEL



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Discussion.

From our simple validation exercise we can see that purchasing an average price 22" colour television set becomes more economical than renting after about 3 years (3 years 3 months). For a cheaper television set the corresponding figure is about 2 years and 7 months and for the expensive television set the figure is about 5 years and 2 months.

This corresponds well with the estimates quoted in 'Which' magazine. 'Buying your own T.V. set makes as much sense economically as we said two years ago. Even if you're tempted by the cost-free repairs you get when you rent, buying a set and getting a maintenance contract would be cheaper after 3 years.....' ('Which', January 1975).

It is clear from the graph that the slope of the purchase cost line is dependent on the value of M, i.e. the smaller M is then the less the slope of the line.

It is possible to purchase a maintenance agreement for the television tube only, for a cost of about £20. Since the tube is the most expensive single component in the television set and the one which is more likely to need replacing, perhaps we should consider the effect of a cheaper insurance on an average priced television set of £300. We could also consider, in view of the better reliability of televisions, the possibility of no maintenance contract.

Let $P = 300$ but $M = 20$ (tube only)

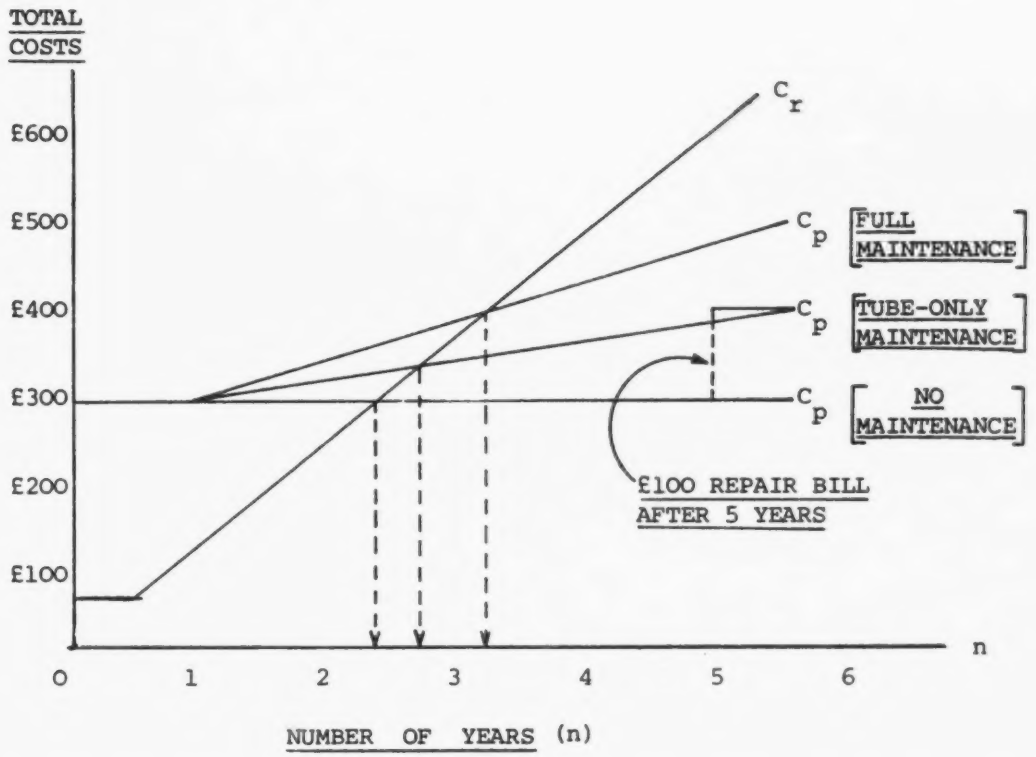
$C_p = 300 + 20(n - 1)$

$C_p = 280 + 20n \quad (n \geq 0)$

Let $P = 300$ but $M = 0$ (no maintenance)

$C_p = 300$

Fig 6. Cost with varying maintenance agreement.



We can see clearly that a reduction from a full maintenance agreement to a 'tube only' agreement decreases the break-even point by approximately six months and having no agreement reduces the break-even point by a further 3 months. Clearly the prospective purchaser will have to weigh up the risks himself. If we assume a major bill of say £100 for the tube after 5 years, it is still cheaper to have no maintenance agreement. Only if there is a major repair bill of say £100 before 3½ years is the customer likely to regret not taking out a maintenance agreement.

5. MODEL DEVELOPMENT ('O' level)

So far we have discussed a fairly simple situation where the prospective purchaser has the necessary cash with which to purchase the 22" colour T.V. set. We now consider a more realistic situation where the prospective customer does not have, or does not wish to use, the necessary cash to purchase the television outright. For this model we will make further assumptions:

There is sufficient money to pay the initial prepayment and monthly payments.

The balance of the purchase price is borrowed from the bank or H.P. company.

The rate of interest is quoted as simple interest rates. (which of course is unrealistic).

We are still left with the original question - should we rent or buy ?

Cost.

The rental costs will remain as before, i.e.

$$C_r = 12nR$$

Let P be the purchase price.

Let x be the fraction of the purchase price required as a prepayment (expressed as a fraction).

Let I be the interest rate (expressed as a fraction) per year.

Let y be the number of years over which money is borrowed.

Let M be the cost per annum of the maintenance agreement.

The total purchase price over n years will be given by:

C_p = Prepayment + monthly repayments + interest + maintenance costs.

$$C_p = xP + (1 - x)' P + yI\ (1 - x)\ P + (n - 1)M$$

$$C_p = P\left[1 + yI(1 - x)\right] + (n - 1)M. \quad \text{(only true if } n > y\text{)}$$

Validation.

Consider a £300 television set on which the deposit required is 20% (Correct at April 1982) and the interest rate is 20% (varies as to the source of the loan). Let the money be borrowed for two years and maintenance be £40 per annum.
Initially £60 is paid as deposit, so £240 needs to be borrowed. At the end of the first year, half the principal (£120) is repaid and interest on the amount borrowed (£48), making a cost in the first year of £168. The outlay in the second year is the same repayment and interest (£168) plus a maintenance charge of £40 i.e. an outlay of £208 and a total outlay in two years of £436. (The interest paid is unrealistically high).

Doing a similar calculation for a £250 and a £450 television set, we get the following table, plotted in Fig 7.

n	0	1	2	3	4	5
C_p	60	228	436	476	516	556
$C_{p.min}$	50	190	370	410	450	490
$C_{p.max}$	90	342	634	674	714	754
C_r	40	120	240	360	480	600

Comparing Figs 5 and 7, we can see how borrowing has significantly increased the break-even time.

Break-even point

	Cash	2 years at 20% interest
$C_{p.min}$	2 years 7 months	3 years 8 months
C_p	3 years 3 months	4 years 6 months
$C_{p.max}$	5 years 2 months	7 years 0 months

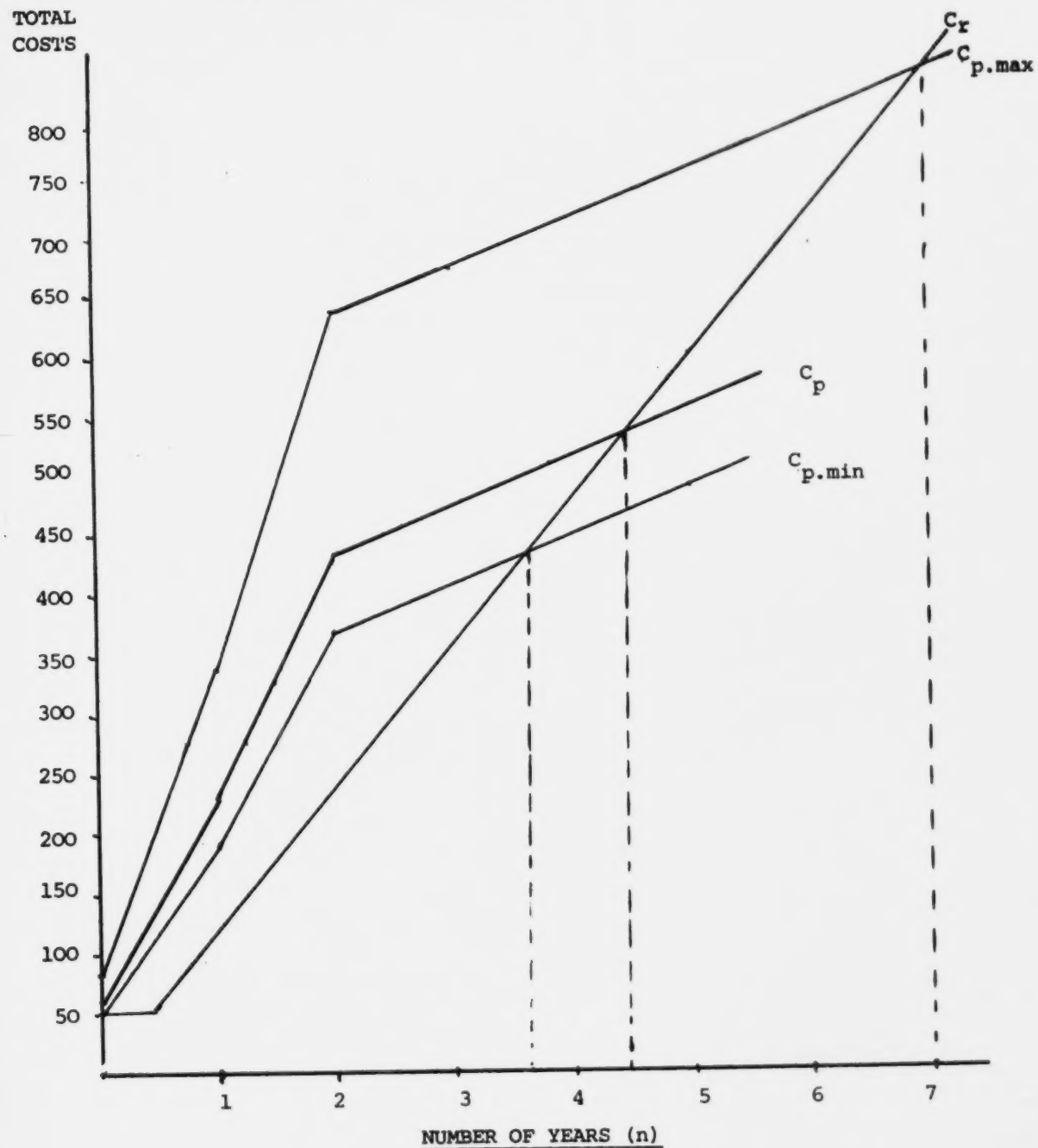
The slopes of the initial parts of the purchase cost lines are dependent on two factors:

(a) interest rate (I)

(b) number of years of borrowing money (y)

At this point a consideration should be made of the effect of varying these two factors.

Fig 7. Costs for varying priced T.V. sets on a two-year loan at 20% simple interest per annum.



Using $C_p = P\left[1 + yI(1 - x)\right] + (n - 1)M$, for $n > 2$

Let P = 300

Let y = 2 years

Let M = 40

Let I vary in 5 stages from 15% to 30%
(15% could well be the lower limit for a bank loan and 30% could well be the top limit for a 'not so reputable' finance house).

2 year loan							
n	0	1	2	3	4	5	6
C_p(15%)	60	216	412	452	492	532	572
C_p(20%)	60	228	436	476	516	556	596
C_p(25%)	60	240	460	500	540	580	620
C_p(30%)	60	257	494	534	574	614	654

The results are plotted on Fig 8 and indicate that a change in interest rate from 15% to 30% on a 2 year loan can extend the break-even point by about one year. (in fact there is slight variation in the slope between years 1 and 2).

1 year loan							
n	0	1	2	3	4	5	6
C_p(15%)	60	336	376	416	456	496	536
C_p(20%)	60	348	388	428	468	508	548
C_p(25%)	60	360	400	440	480	520	560
C_p(30%)	60	372	412	452	492	532	572

The table above considers a loan made over only one year, the results are plotted in Fig 9.

FIGURE 8 Varying interest rates for a 2 year loan

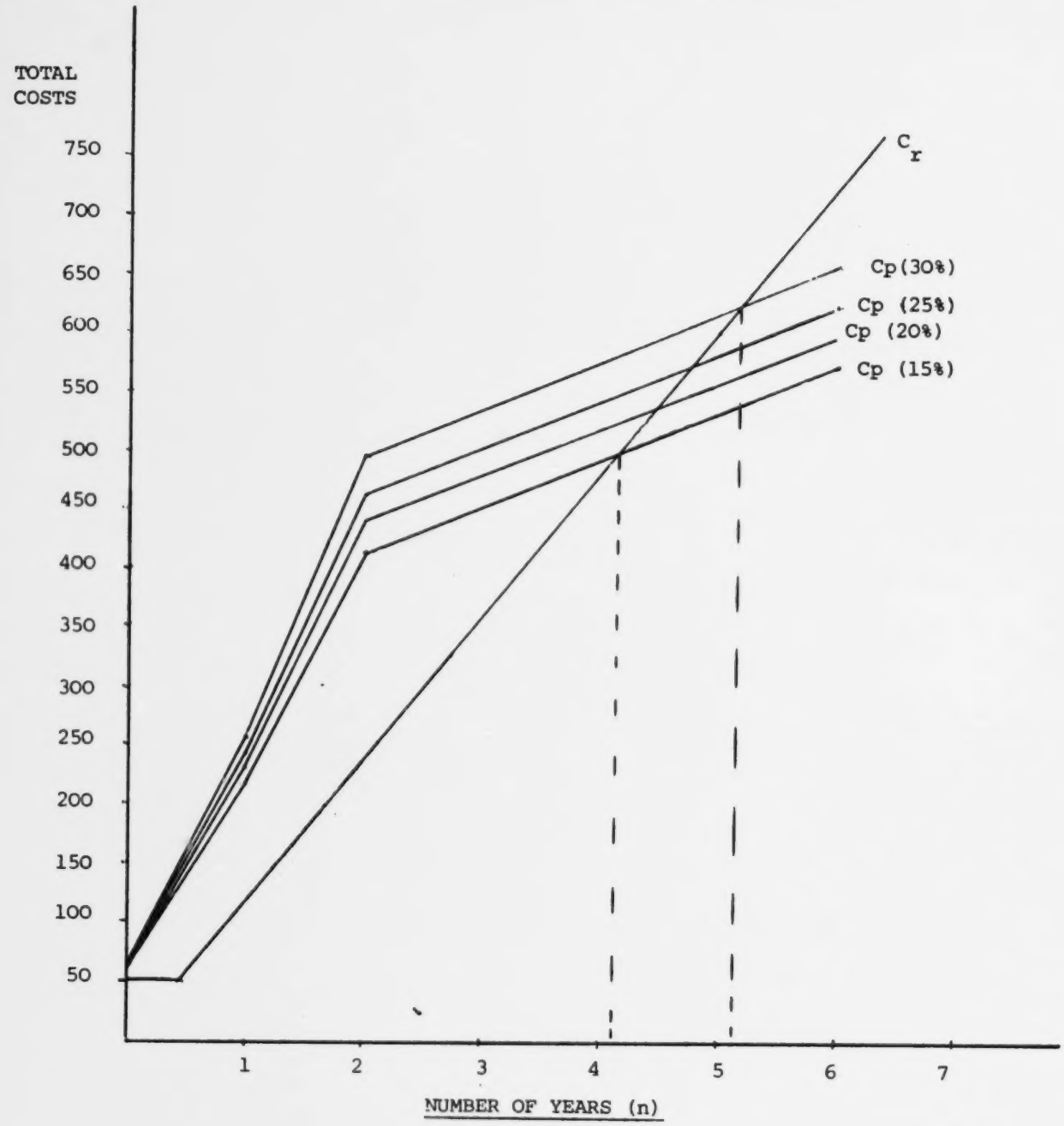
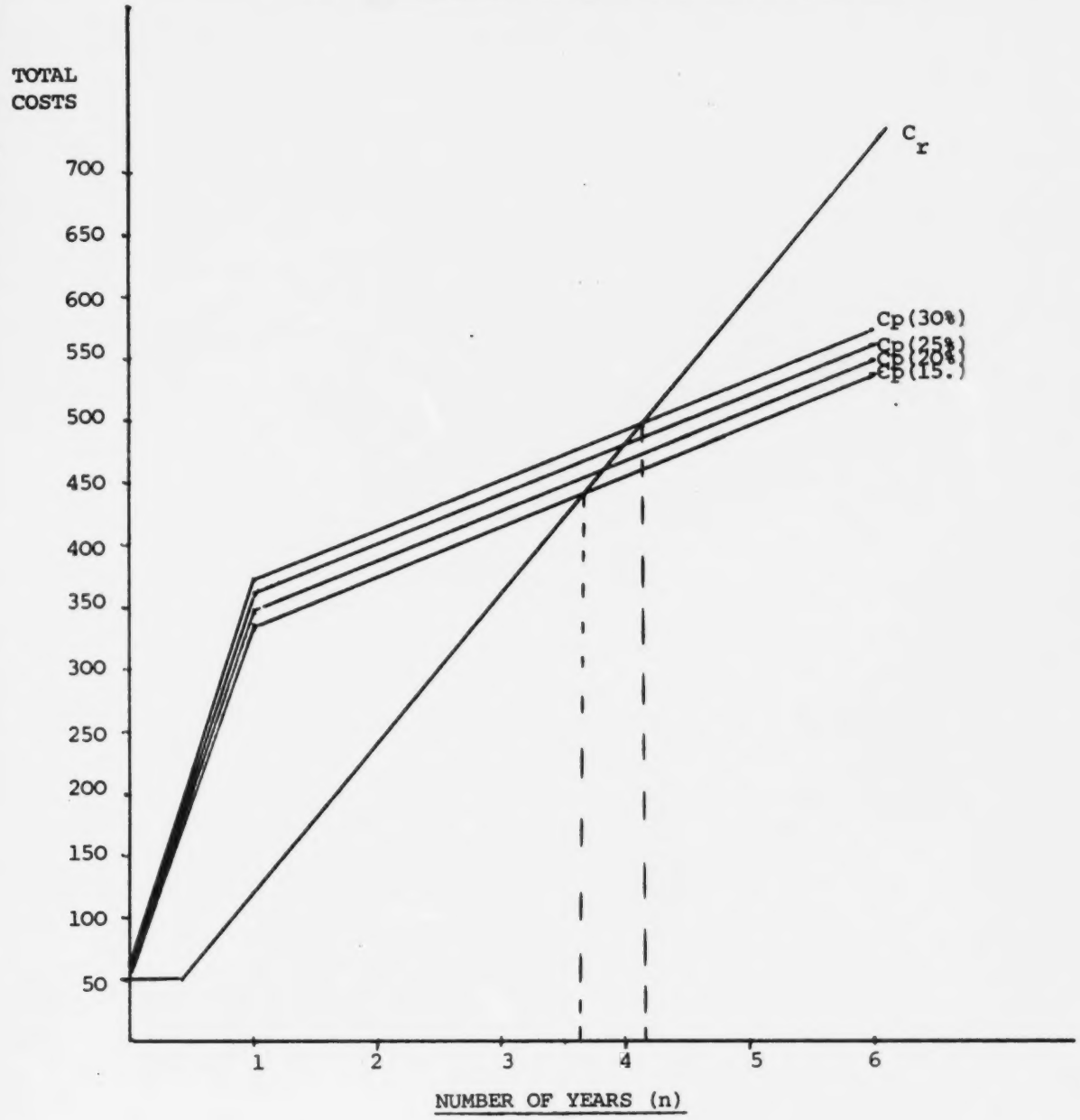


FIGURE 9 Varying interest rates for a 1 year loan



The results in Fig 9 indicate that a change in interest rates from 15% to 30% on a 1 year loan can extend the break-even point by about 6 months.

We can see that the change in break-even point from a 1 year loan at 15% to a 2 year loan at 30% is from 3 years 3 months to 5 years 2 months, clearly indicating the necessity for shopping around to obtain the cheapest interest rates available.

But since an average colour television set has an expected lifetime of something approaching 10 years, it would appear to be beneficial to purchase rather than rent, even though our more sophisticated model has extended the break-even point for an average-priced 22" colour television from 3 years 2 months with immediate payment in full to 4 years 6 months with a 20% deposit and the balance paid over 2 years.

Again the model produces figures which are still in line with the 'Which' report that even buying your television set in the most expensive way could save you money over renting in about 6 years. But their figure includes paying the full list price for the television set which would make the price more than our £300 and consequently more interest would need to be paid.

6. MODEL DEVELOPMENT ('A' LEVEL)

To extend the model further it would now be suitable to:

(a) introduce the effect of increasing the maintenance cost as the age of the set increases.

(b) take into account the effect of compound interest on the loan.

(c) to make a consideration for the people who borrow money from a finance house over a longer period than the normal two years and pay compound interest on the loan.

Age-linked maintenance agreement

We require that $M = f(n)$ such that M is zero for the first year and increases in subsequent years. M is zero for the first year because the set is covered by the guarantee during the first year.

A number of possible functions could be used. An interesting exercise would be to formulate a sub-model for age-linked maintenance. Alternatively an arbitrary function can be specified and worked out to see what kind of results can arise. We assume that the initial payment is £40 and increases according to the function below.

$$M = M_1 \left(1 + \frac{(n-1)}{10}\right) \text{ for } n > 0$$

$$M = 0 \quad \text{for } n = 0$$

The total maintenance costs for n years will be given by ΣM.

$$\Sigma M = 0 + \sum_{i=1}^{n} M_1 \left(1 + \frac{i-1}{10}\right)$$

$$= nM_1 + \frac{M_1}{10} \cdot \frac{n(n-1)}{2}$$

$$= \frac{M_1}{20} n(n+19)$$

$$\text{Total maintenance Cost} = \frac{M_1}{20}(19n + n^2)$$

This is calculated in the table below, which shows that the function is rather unrealistic. (it would be cheaper to buy a new television every six years).

Year	Fixed price contract	'Age-linked' agreement
0	£ 0	£ 0
1	40	40
2	80	84
3	120	132
4	160	184
5	200	240
6	240	300
7	280	364
8	320	432
9	360	504
10	400	580

Compound Interest

Let P = Purchase price

Let x = fraction of purchase price required as a prepayment

Let I = compound interest rate per year

Let y = number of years of the loan

Prepayment = Px

Amount borrowed = $P(1 - x)$

If the balance is paid at the end of the loan period

Amount repaid = $Px + P(1 - x)(1 + I)^y$

Maintenance term = $(n - 1)M$

$$C_p = Px + P(1 - x)(1 + I)^y + (n - 1)M$$

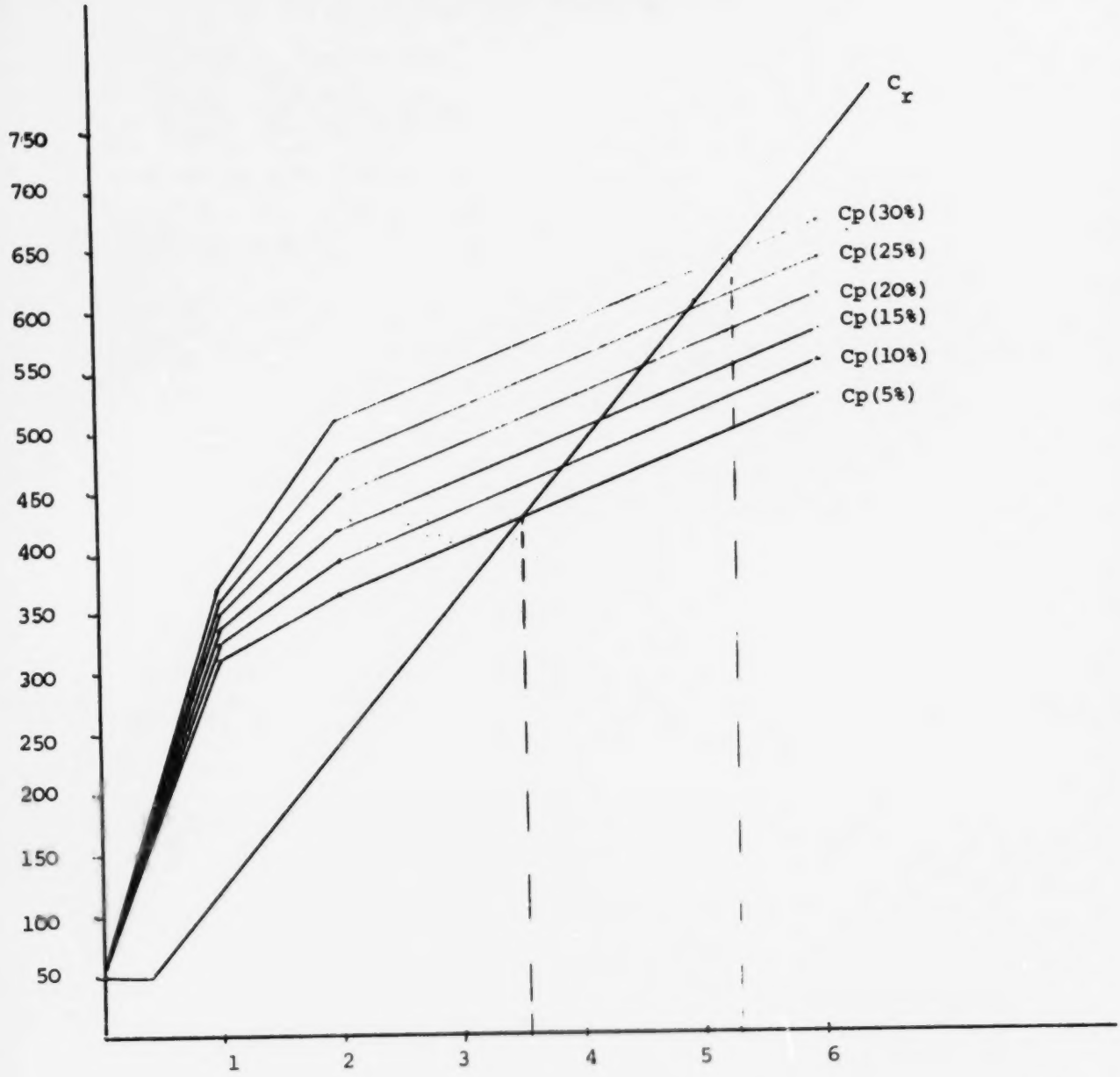
If we take P = £300, x = 1/5, M = 40 per annum, (i.e. non-age-linked agreement), y = 2. Fig 10 shows the results obtained by taking different values of I from 5% up to 30% with a two year loan period, y = 2. Clearly a longer loan period would make purchase less favourable. The 'break-even point is increased from 3 years 8 months (5%) to 5years 5 months (30%).

n	0	1	2	3	4	5	6
Cp (5%)	60	312	365	405	445	485	525
Cp (10%)	60	324	390	430	470	510	550
Cp (15%)	60	336	417	457	497	537	577
Cp (20%)	60	348	446	486	526	566	606
Cp (25%)	60	360	475	515	555	595	635
Cp (30%)	60	372	506	546	586	626	666

The model of compound interest is fairly simplistic in that it ignores the fact that payments are usually made at regular intervals rather than all together at the end of the loan period. It can be shown, if we assume monthly payments with compound interest added monthly, the following total repayment is obtained.

$$\text{Total Repayment} = yIP(1 - x)\left[1 - \left((1 + \frac{I}{12})^{12y}\right)^{-1}\right]^{-1}$$

FIGURE 10 Varying compound interest rates



7. CONCLUSION.

These models show how a chosen situation can be modelled at various levels of sophistication according to the age and ability of the students being taught. The initial model is a straightforward comparison of cost, whilst later work incorporates how interest rates or loans may effect the decision. The final model using compound interest could be linked to the method for calculating repayments on a mortgage. The data included for validation relates to 1982 but prices of televisions and renting have not increased substantially. Basically each of the models show the advantage of purchasing over renting. Moreover, as televisions have now become much more reliable, purchasing seems to be even more sensible. A new market is the renting of video-recorders which could be modelled; there is an extra factor of the video-cassettes themselves.

III TRAVEL TIME TO SCHOOL

Gill Close

1. INTRODUCTION

There are not many models in the literature suitable for average pupils, and one motivation for this model was to find a problem that could be used in the classroom with such pupils.

In the south part of a London Borough the problem of falling rolls is seriously affecting three local comprehensive schools A, B, C. There is much discussion at local government, union and parental level as to the possible policies and their consequences. It is likely that one school will be shut down and its pupils redistributed between the remaining two schools.

A worry which has been voiced by parents is the extra time which pupils will spend in travelling to an alternative school, especially during the short days in winter. The problem of travel time to a school is just one example of the general problem of siting public amenities.

It is proposed to close down school A and to redistribute the pupils equally between schools B and C. A model must be built to determine if the pupils will need more time to reach school B or C than school A and which of schools B and C will be quicker for them to reach.

Learning Aims

This work serves not only to introduce aspects of modelling but to reinforce other mathematical concepts in a concrete context. The specific ideas studied are:

1. A model as a reasonable simplification of reality.
2. Validation.
3. The use of data and methods for its collection.
4. The use of variables.
5. Decision variable, environmental parameters and decision criteria.
6. Sensitivity analysis.
7. Influence diagrams.
8. The interpretation of a formula and negative quantities.
9. Decimals, percentages, proportion, estimation, inequalities, reading tables, scales, using maps, straight line graphs and manipulating algebraic expressions in a realistic context.

Age-range: 11-16 year old pupils

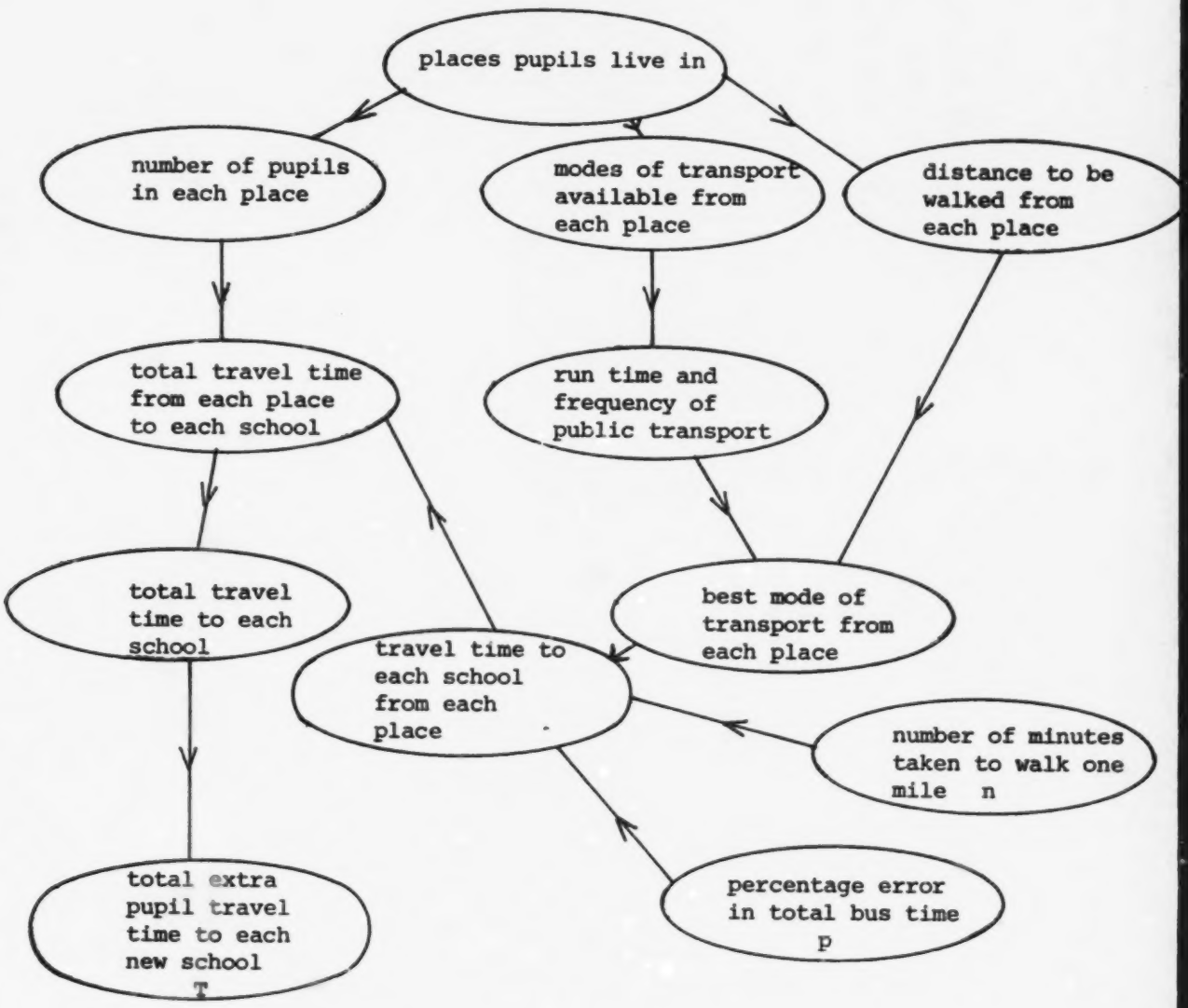
2. ASSUMPTIONS

1. Pupils are considered as living at selected points.
2. Those pupils living very far away would be allocated to a school nearer their home.
3. Ignore the small number of pupils who come by car.
4. Public transport is used for distances over 0.6 miles when it is convenient.
5. Where two buses follow a similar route the average before and after walking distances are used.
6. Always choose the quicker form of public transport - ignore cost considerations
7. Take underground and British Rail journey times as accurate with no waiting time assumed.
8. Take the average waiting time for a bus to be 3 minutes.
9. The error in the total bus travel time is proportional to the total bus travel time. Let p be the percentage error.
10. Assume pupils all walk at the same speed. They take n minutes to walk one mile.
11. The decision criterion T will be 'total extra pupil travel time' between one school and another.

Assumption 1 means that we ignore the fact that children's homes are spread through the borough. The assumption of a number of discrete points where homes are concentrated is done to make the model easy. Also, since many of the children come from high-density estates the assumption is fairly reasonable. The model could be modified by using a greater or smaller number of points. Assumption 2 is a minor one and results in the deletion of about 30 pupils from outside the borough who will be reallocated to schools in their own borough. Assumption 3 again deletes a minor group of pupils. Assumption 4 gives a criterion for deciding when someone would choose to come to school by public transport rather than walk. This assumption could also be modified to give a different criterion and there is scope here for pupils to investigate and collect data to try to decide what a good value for this distance would be. This shows the need which sometimes arises, for the collection of data to make our assumptions as good as possible. Assumption 5 deals with the situation where more than one bus route is available to pupils from one point, with maybe different walking distances involved. It seems reasonable to average the different distances. Assumption 6 could be highly contentious, but if we did not assume this we would have to develop a model for the effect of the different cost of different modes of travel, and this assumption in such a model may be difficult to justify. Certainly, if the group feels very strongly that cost is a major factor, this could be modelled by changing assumption 6. As far as assumption 7 is concerned this is very much a pious hope! Again, however, we need to take some decisions to set up our model, and, in the light of experience, we may wish to amend this assumption and include some waiting time for tube and train. Assumption 8, like number 4, is another one where the opportunity exists for pupils to collect and examine data to test the validity of the assumption, and if necessary change it. Many models of situations where waiting is involved use simulation methods based on the assumption that waiting times follow a particular statistical distribution with given mean and variance. The same approach could be used in this model, but it would extend the work to a more advanced level. However, this does open up the possibility of pupils modelling aspects of the same problem at different stages in their school career, using more sophisticated techniques as they learn more mathematics.

We do need to mirror the fact that bus times will vary from day to day because of the general traffic conditions, and assumption 9 is one possible way of dealing with this. Assumption 10 is obviously not true, but is the simplest thing to assume to get the model off the ground. Once again, pupils can carry out surveys to find a reasonable average walking speed, and also, with more advanced students different possible distributions of speeds can be used. An important point is that it may be reasonable to assume that walking is related to age of pupil, and data could be examined to bring this into the model. The last assumption gives us a decision criterion which it is the purpose of the model to produce.

3. INFLUENCE DIAGRAM


places pupils live in
number of pupils in each place
modes of transport available from each place
distance to be walked from each place
total travel time from each place to each school
run time and frequency of public transport
total travel time to each school
best mode of transport from each place
travel time to each school from each place
number of minutes taken to walk one mile n
total extra pupil travel time to each new school T
percentage error in total bus time p

4. MODEL DEVELOPMENT

The model was constructed here without undertaking any surveys of the positions of pupils' homes as exact data was unavailable. For use with pupils, however, the homes of either a random sample of perhaps 100 pupils or in fact of every pupil could be marked on a map of the area and pupils be questioned as to their favoured mode of transport to school. A survey could also be made of the times taken for each of these journeys. The pupils' average walking speed could also be estimated empirically.

The first step taken in constructing the model was to draw an outline map of the area. It was noted that the schools were very near the edge of the borough and therefore would not draw pupils from any symmetrically placed regions. Because of the great concentration of pupils in certain council estates or densely populated areas, it was felt it would be sufficiently accurate to consider them as living at points q to z, the centres of each of these regions.

At this stage all factors influencing travel time were considered, a rough influence diagram was drawn up and environmental parameters were defined. The exact decision criterion and decision parameters could not be formulated until more details had been considered.

Experienced pastoral staff of school A identified the appropriate places in which pupils' homes were concentrated and were asked to give the proportion of pupils coming from each region. There are 1,000 pupils attending school A, of whom 30 are from outside the borough or a great distance away. These pupils would be assigned to a school in their borough or nearer to their home. The remaining 970 pupils have been allocated to each place in accordance with the proportions elicited from pastoral staff. This information is shown on the outline map.

From each place appropriate modes of transport were worked out using bus and train maps. These are also illustrated on the outline map. No children come to school by bicycle and the number being brought by car is so small that it was neglected. It was assumed that very convenient public transport would always be used in preference to walking distances of over 0.6 miles. Real walking distances for complete journeys and with using public transport were measured from the map and listed in Table 1, together with the type of public transport used. Where there was a choice of two buses following almost the same route the average before and after walk distances were used. Only one method of travel from each place to each school was listed. Where two methods using public transport existed the quicker method was listed.

Public transport fell into three categories : bus, underground train and British Rail train. Both the underground and British Rail trains run to a published timetable from which journey times were calculated. It was also assumed that as these were fixed pupils would arrive at stations in time to catch specific trains each day. London Transport provided details of bus frequencies and average journey times. Using this information the average waiting time for each bus was taken as 3 minutes. It was assumed that the time lost or gained relating to traffic density and bunching would be proportional to the total bus travel time. The total bus travel time for each journey was calculated as the average journey time + 3 minutes + a percentage (p) of this total time, e.g. a timetabled journey quoted as taking 10 minutes would have a total bus travel time of 13+13p minutes. All of the times relating to public transport were based on journeys to school between 8.00 a.m. and 8.45 a.m.

OUTLINE MAP SHOWING SCHOOLS A,B AND C, BUS AND TRAIN ROUTES AND DISTRIBUTION OF HOMES OF SCHOOL A PUPILS

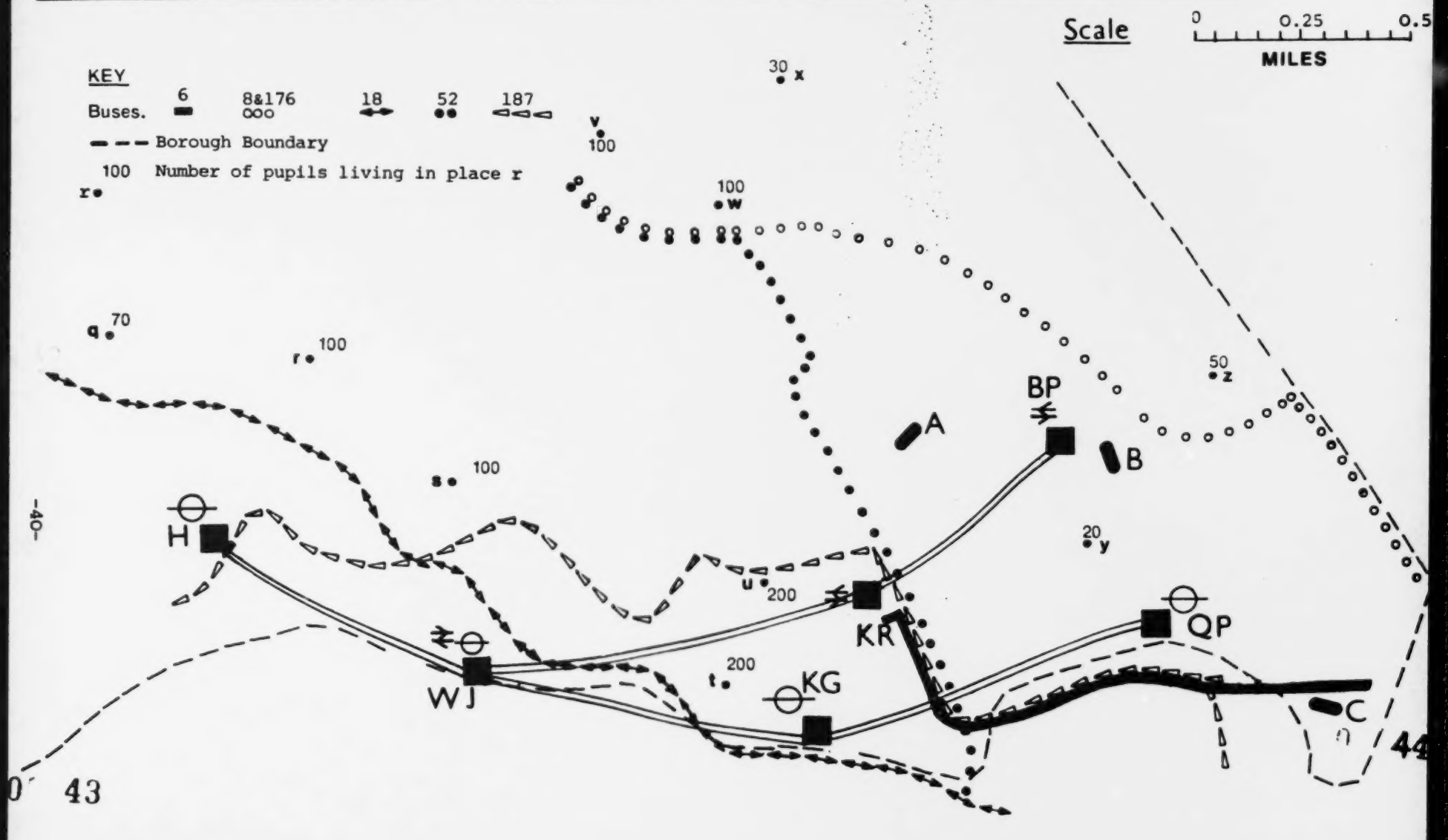


TABLE 1 METHOD OF TRAVEL FROM EACH PLACE TO EACH SCHOOL

PLACE	SCHOOL A	B	C
q	0.1 18 0.85	0.5 H-W.J-B.P	0.5 H-Q.P 0.4
r	0.25 187 0.25	0.4 H-Q.P 0.4	0.3 187 0.2
s	1.5	0.4 W.J-B.P	0.2 187 0.2
t	0.85	1.2	0.25 6,187 0.1
u	0.6	1	0.1 6,187 0.1
v	0.1 52 0.25	0.1 8,176 0.1	0.1 8,176 0.4
w	0.8	0.1 8,176 0.1	0.1 8,176 0.4
x	1.1	0.5 8,176 0.1	0.5 8,176 0.4
y	0.6	0.3	0.7
z	1	0.5	1.1

Key : 0.1 18 0.85 means a walk of 0.1 miles from q, followed by bus 18 and a walk of 0.85 miles to school A.
H, W.J, K.G, Q.P, K.R, B.P are train stations.
1.5 is the distance walked from s to school A.

All pupils were assumed to take n minutes to walk one mile.

Using this data and these assumptions Table 2 was prepared to show the time taken in minutes for travel from each place to each school.

As a measure of pupils' extra time to reach their new school it was decided to consider the total extra time taken for all of school A's pupils to travel to school B or C rather than to school A. All, rather than half, of the pupils were considered so that alternative methods of allocating the pupils could be considered later using the same calculations. This total extra time was found by summing the extra time taken for all pupils living in each place. The calculation was made like this in order that direct comparison of extra travel times to each school for pupils from each place could be made later if desired. The difference in total travel time to school B and school C, T_{C-B}, was also calculated in order to determine which one of schools B and C it would be quicker for school A pupils to reach.

These assumptions and calculations led to the choice and naming of two decision variables, p and n, and the decision criterion T shown in the influence diagram.

TABLE 2 TIME TAKEN IN MINUTES FROM EACH PLACE TO EACH SCHOOL

Key n = number of minutes taken to walk each mile
p = percentage error in bus journey time
(bus times include journey time and waiting time)

PLACE	SCHOOL		
	A	B	C
q	0.95n + 14+14p	0.5n + 9	0.9n + 7
r	0.5n + 13+13p	0.8n + 7	0.5n + 18+18p
s	1.5n	0.4n + 4	0.4n + 15+15p
t	0.85n	1.2n	0.35n + 7+7p
u	0.6n	n	0.2n + 9+9p
v	0.35n + 13+13p	0.2n + 11+11p	0.5n + 15+15p
w	0.8n	0.2n + 7+7p	0.5n +11+11p
x	1.1n	0.6n + 6+6p	0.9n +10+10p
y	0.6n	0.3n	0.7n
z	n	0.5n	1.1n

The total extra time taken to travel to school B rather than school A, T_{B-A}, is calculated by summing the extra time for each place.

$$
\begin{aligned}
T_{B-A} = \quad & 70\,[(0.5n + 9) - (0.95n + 14 + 14p)] \\
& + 100\,[(0.8n + 7) - (0.5n + 13 + 13p)] \\
& + 100\,[(0.4n + 4) - (1.5n)] \\
& + 200\,[(1.2n - 0.85n)] \\
& + 200\,(n - 0.6n) \\
& + 100\,[(0.2n + 11 + 11p) - (0.35n + 13 + 13p)] \\
& + 100\,[(0.2n + 7 + 7p) - (0.8n)] \\
& + 30\,[(0.6n + 6 + 6p) - 1.1n] \\
& + 20\,(0.3n - 0.6n) \\
& + 50\,(0.5n - n)
\end{aligned}
$$

$$
\begin{aligned}
T_{B-A} = \quad & 70\,(-0.45n - 5 - 14p) \\
& + 100\,(0.3n - 6 - 13p) \\
& + 100\,(-1.1n + 4) \\
& + 200\,(0.35n) \\
& + 200\,(0.4n) \\
& + 100\,(-0.15n - 2 - 2p) \\
& + 100\,(-0.6n + 7 + 7p) \\
& + 30\,(-0.5n + 6 + 6p) \\
& + 20\,(-0.3n) \\
& + 50\,(-0.5n)
\end{aligned}
$$

$$
\begin{aligned}
T_{B-A} = \quad & -31.5n - 350 - 980p \\
& + 30n - 600 - 1300p \\
& - 110n + 400 \\
& + 70n \\
& + 80n \\
& - 15n - 200 - 200p \\
& - 60n + 700 + 700p \\
& - 15n + 180 + 180p \\
& - 6n \\
& - 25n
\end{aligned}
$$

$$T_{B-A} = 130 - 82.5n - 1600p$$

The total extra time taken to travel to school C rather than to school A, T_{C-A}, is calculated by summing the extra time for each place.

$$
\begin{aligned}
T_{C-A} = \quad & 70 \left[(0.9n + 7) - (0.95n + 14 + 14p)\right] \\
& + 100 \left[(0.5n + 18 + 18p) - (0.5n + 13 + 13p)\right] \\
& + 100 \left[(0.4n + 15 + 15p) - 1.5n\right] \\
& + 200 \left[(0.35n + 7 + 7p) - 0.85n\right] \\
& + 200 \left[(0.2n + 9 + 9p) - 0.6n\right] \\
& + 100 \left[(0.5n + 15 + 15p) - (0.35n + 13 + 13p)\right] \\
& + 100 \left[(0.5n + 11 + 11p) - 0.8n\right] \\
& + 30 \left[(0.9n + 10 + 10p) - 1.1n\right] \\
& + 20 \quad (0.7n - 0.6n) \\
& + 50 \quad (1.1n - n)
\end{aligned}
$$

$$
\begin{aligned}
T_{C-A} = \quad & 70 \quad (-0.05n - 7 - 14p) \\
& + 100 \quad (5 + 5p) \\
& + 100 \quad (-1.1n + 15 + 15p) \\
& + 200 \quad (-0.5n + 7 + 7p) \\
& + 200 \quad (-0.4n + 9 + 9p) \\
& + 100 \quad (0.15n + 2 + 2p) \\
& + 100 \quad (-0.3n + 11 + 11p) \\
& + 30 \quad (-0.2n + 10 + 10p) \\
& + 20 \quad (0.1n) \\
& + 50 \quad (0.1n)
\end{aligned}
$$

$$
\begin{aligned}
T_{C-A} = \quad & -3.5n - 490 - 980p \\
& \qquad + 500 + 500p \\
& - 110n + 1500 + 1500p \\
& - 100n + 1400 + 1400p \\
& - 80n + 1800 + 1800p \\
& + 15n + 200 + 200p \\
& - 30n + 1100 + 1100p \\
& - 6n + 300 + 300p \\
& + 2n \\
& + 5n
\end{aligned}
$$

$$T_{C-A} = 6310 - 307.5n + 5820p$$

Sensitivity Analysis

Sensitivity analysis can be carried out on T_{B-A} and T_{C-A} by estimating values for n and p. A walking speed of 3mph (i.e. 20 min per mile)and an error of 10% extra in total bus time are reasonable estimates, giving n=20 and p=0.1.

Using these values $T_{B-A} = 130 - 82.5x20 - 1600x0.1$
$= \underline{- 1680}$

This indicates that it is quicker to travel to school B than to school A by a total of 1680 minutes or, averaging amongst 970 pupils, by about 1.7 minutes per pupil.

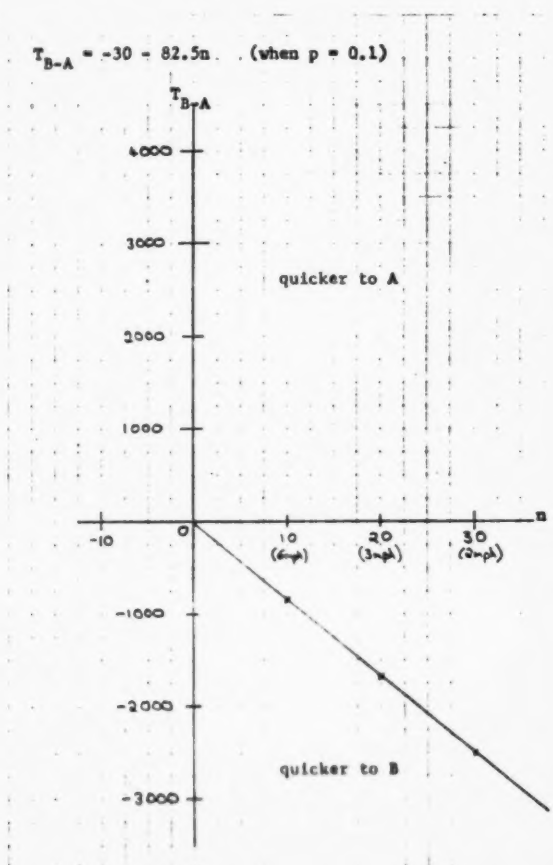
The proportion of the actual journey time which 1.7 minutes represents can be ascertained separately for each place or after calculation of a weighted average journey time to school A. It is valuable to discuss here the minimum absolute time saved or proportion of time saved which would be likely to affect a parent's choice of school.

To analyse the effect on T_{B-A} of changes in the estimate of n, we can assume p=0.1 and rewrite the equation as $T_{B-A} = 130 - 82.5n - 160$
$= \underline{-30 - 82.5n}$

A graphical representation of this aids interpretation and requires discussion of appropriate and inappropriate values of n, including n = 0, and the speeds they represent. As a calculation of T_{B-A} is made for each point plotted the average walking time per pupil (t) can be found by dividing this value by 970. Alternatively, using the fact that each 1000 minutes of T_{B-A} is approximately 1 minute per pupil can put the magnitude of the time differences into perspective.

$T_{B-A} = -30 - 82.5n$

n	T_{B-A}	t
10	-855	-0.9
20	-1680	-1.7
30	-2505	-2.6



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It can be seen that it is always quicker to walk to B than to A and that, the slower a pupil walks, the bigger is the time difference between their journey to B and to A although, clearly, their journey to B is not actually quicker than it would be if they walked more quickly.

Algebraic methods yield a break-even point in journey times to B and to A when $T_{B-A} = 0$ i.e. $-30 - 82.5 = 0$

$82.5n = -30$

$n = -0.4$

Journeys that are quicker to B than to A occur when $T_{B-A} < 0$

i.e. $-30 - 82.5n < 0$

$82.5n > -30$

$n > -0.4$

As negative values of n have no meaning in this context it follows that it will always be quicker overall to travel to school B than to school A, whatever values are taken for n.

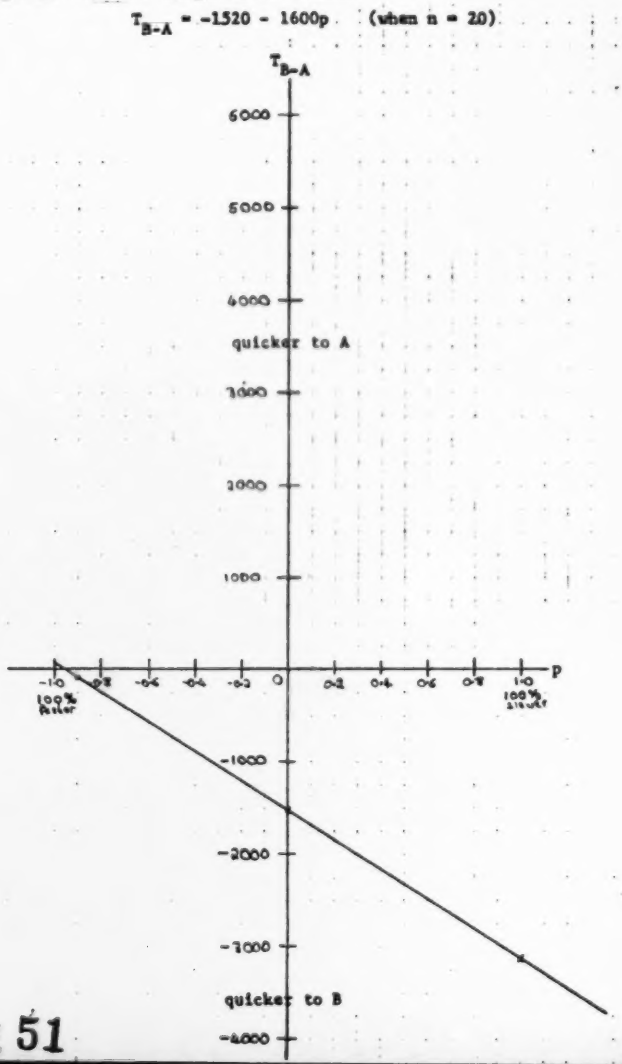
To analyse the affect on T_{B-A} of changing p we rewrite the equation for T_{B-A} using n = 20 to obtain: $T_{B-A} = 130 - 82.5 \times 20 - 1600p$

$= \underline{-1520 - 1600p}$

In selecting appropriate values of p for plotting a graph, decimals will need to be converted to percentages and the meaning of values of p: $-1 \leq p \leq 1$ discussed. It will be seen that p differs from n in that some negative values for p, but not all, have a meaning in this context for example when p = -1 a bus will arrive in zero time and for $p < -1$, in negative time.

$T_{B-A} = -1520 - 1600p$

p	T_{B-A}	t
1	-3120	-3.2
0	-1520	-1.6
-0.9	-80	-0.1



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From the graph it can be seen that, for all sensible estimates of p it is quicker overall to B than to A. It is also clear that, as bus journeys became slower than timetabled the difference between journey times to B and to A increases.

At break-even point $T_{B-A} = 0$ i.e. $-1520 - 1600p = 0$

$1600p = -1520$

$p = -0.95$

In other words journey times to A and B are equal when buses are 95% faster than advertised i.e. they take only 5% of their advertised time. This is a highly unlikely occurence. Journeys are quicker to A than B when $T_{B-A} > 0$

i.e. when $p < -0.95$. Consequently for all reasonable estimates of p journey times are quicker to B than to A.

Thus any reasonable estimates of n and p will result in the overall journey times to school B being less than those for going to school A. From the graphs it can be seen that T_{B-A} is not sensitive to changes in n or p.

For T_{C-A} where $T_{C-A} = 6310 - 307.5n + 5820p$, estimating n and p as 20 and 0.1 respectively gives $T_{C-A} = 6310 - 6150 + 582$

$= \underline{742}$

This means that it is quicker to travel to A than to C by 742 minutes or 0.8 minutes per pupil on average.

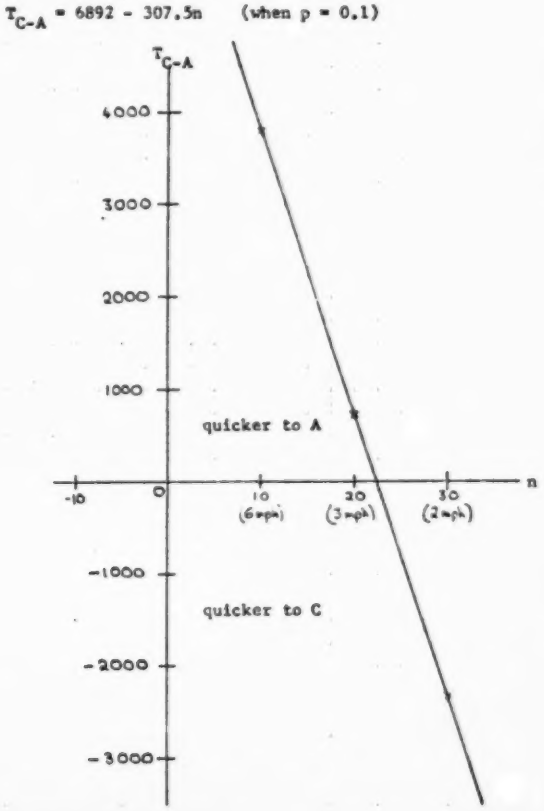
To analyse the effect on T_{C-A} of changing n we estimate p as 0.1 and rewrite the equation as: $T_{C-A} = 6310 - 307.5n + 582$

$= \underline{6892 - 307.5n}$

Plotting this graph on the same scale as that of T_{B-A} against n enables comparison.

$T_{C-A} = 6892 - 307.5n$

n	T_{C-A}	t
10	3817	3.9
20	742	0.8
30	-2333	-2.4



Break-even point occurs where $T_{C-A} = 0$ i.e. $307.5n = 6892$

$n = 22.4$

This represents a speed of 2.7mph.

For speeds quicker than 2.7mph (i.e. $n < 22.4$) it is quicker to school A and for speeds slower than 2.7mph (i.e $n > 22.4$) it is quicker to school C. The model is clearly highly sensitive to changes in n.

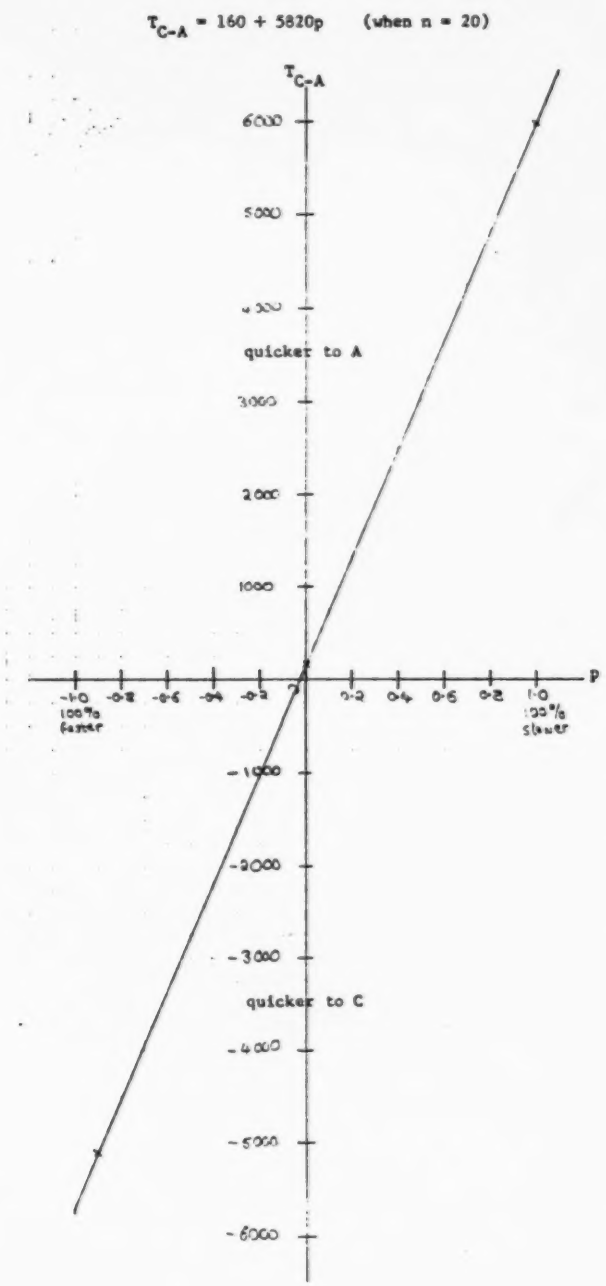
To analyse the effect on T_{C-A} of changing p we estimate n as 20 and rewrite the equation as: $T_{C-A} = 6310 - 6150 + 5820p$

$= \underline{160 + 5820p}$

Plotting this graph on the same scale as that of T_{B-A} against p enables comparison.

$T_{C-A} = 160 + 5820p$

p	T	t
1	5980	6.2
0	160	0.2
-0.9	-5078	-5.2



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Break-even point occurs when $T_{C-A} = 0$ i.e. $5820p = -160$

$p = -0.027$

This represents a bus journey time 2.7% faster than advertised.

For average bus journey times more than 2.7% faster than advertised, journeys to C will be quicker than those to A. For average bus journey times slower than this it will be quicker to A. The model is clearly highly sensitive to changes in p.

Thus the difference in journey times to schools C and A depends critically on the estimates of n and p.

Closer analysis of the gradient of the graphs reveals that changes in n have far more effect on T_{C-A} than T_{B-A} and that changes in p have more effect on T_{C-A} than T_{B-A}.

A change of 4 minutes in n or of 0.2 in p results in a change in T_{B-A} equivalent to 0.3 minutes per pupil on average. A change of 4 minutes in n or of 0.2 in p results in a change in T_{C-A} equivalent to approximately 1.2 minutes per pupil on average.

In order to decide which one of the schools B or C it is quicker for pupils to reach T_{C-B} could be obtained region by region or from $T_{C-A} - T_{B-A}$. The equation for T_{C-B} is $T_{C-B} = 6180 - 225n + 7420p$.

With usual values for n and p we obtain $T_{C-B} = 2422$ i.e. it is quicker to travel to B than to C by 2.5 minutes per pupil on average. The break-even point occurs when n = 30, at a speed of 2mph. When $n > 30$ (i.e. for speeds less than 2mph) it is quicker to travel to C, but for the more likely walking speeds of greater than 2mph it is quicker to travel to B.

Working with respect to p we get a break-even point of p = -0.23 which means that it is quicker to C, only if the buses go faster than their advertised timetables by 23%. Again this is fairly unlikely, but not impossible, so most reasonable assumptions give the results that it is quicker overall to get to school B.

Self-assessment Questions

The first self-assessment questions for pupils of this ability should not differ greatly from the model discussed. The following extend the ideas already encountered and relate to the same school closure problem.

1) It is proposed to close down school C and to redistribute the pupils equally between schools A and B. Build a model to determine if pupils will need more time to reach schools A and B than school C. (Note: it is known that most of the pupils from school C live within 0.4 miles of school C and walk to school).

2) School A is to be closed down and the pupils allocated equally between the two other schools on the basis of ease of travelling between their home and the school. Build a model to decide which children to allocate to each school.

3) It is not known whether to close school A or school C. Build a model to determine which closure will bring about the least extra travelling time for pupils.

Question 1 presents a simplified problem with many of the same modelling features. As the pupils mainly walk to school and live in close proximity to it there is a smaller number of regions to be considered.

Question 2 requires a region by region analysis and a choice of decision criteria. If only time is chosen as a criterion then great attention must be paid to estimated values of n and p in deciding which regions give positive or negative values of $T_{C-B.}$ Also 485 pupils are to be allocated to each school, but the number in each region is a multiple of 10; therefore pupils from one of the regions will have to be allocated randomly.

Question 3 gives the opportunity for previous results to be used but 'least extra travelling time' may seem more meaningful only when applied to pupils for whom it makes a large difference i.e. in certain regions only. Experience of results for various decision criteria in the earlier models can be utilised. A choice may be made between considering 'total extra travelling time' for all pupils and a region by region analysis.

5. CONCLUSION

The average time saving per pupil is less important than the time spent on the actual journey and does not reveal if pupils from particular places will incur far longer travel times by changing school. An analysis of extra travel times for pupils from each place would give parents a clearer guideline.

The results, however, should allay fears in general that a great deal of extra travel time will be incurred by allocating school A pupils to school B or C.

The initial problem stated that school A pupils would be allocated equally to schools B and C. The journey times to school C are likely to be greater than those to school B so some acceptable ways in which this allocation might be made are : entirely randomly, by random choice of half the pupils from each region or by selecting pupils from the regions from which travel to school C took the smallest amounts of time longer than travel to school B. The method of allocation has not been discussed in this problem. Accordingly extra travel times have been calculated for the whole school so that any method of allocation could be considered later using these results.

Clearly assumptions about bus delays did not affect results greatly and it therefore seems acceptable to have ignored variations in train times, a far less frequently used mode of transport.

In developing a linear model speed could not be used as a decision variable. Instead n, the number of minutes taken to walk one mile, was used. Frequent conversion of values of n into speeds in miles per hour and vice-versa offers useful practice.

This model enables pupils to use real data to answer a real problem. In interpreting the solutions practice is had in understanding the meaning of negative numbers in a real situation, and also the different effects of varying the parameters is discovered. An important distinction between the two variables is that, in one case negative values are not possible, but in the other they are.

This problem was based on one particular London borough, but the approach can be used in other areas. The model has a great deal of scope for collection of data of various kinds by the pupils themselves and can also be adapted to generate some high level problems, particularly by using statistical distributions and simulation.

IV. OPTIMUM SELLING PRICE

John Messer

1. INTRODUCTION

A shop manager, currently selling a product for the recommended retail price wonders whether reducing the price would yield an increase in sales. What is the optimum selling price that will maximise profits?

This model provides an introduction to the use of modelling as a tool for other disciplines, in this case simple economics.

It also provides an opportunity to apply some of the mathematics previously learned, in an interesting and more useful way than the usual contrived exercises.

Pre-requisites

Construction of equations, algebraic manipulation.

Maximum and minimum values of a function.

Age-range 14 - 18 years.

2. ASSUMPTIONS

Three basic assumptions are required.

1. The cost price includes the cost of handling, storage and marketing.

2. The number of potential sales is not restricted by problems of limited supply.

3. The number of sales depends solely on the selling price.

An initial consideration yields the following fixed values.

r = recommended selling price.

c = cost price.

P = net profit.

N = current sales. (number sold)

P = N (r-c)

However, we wish to consider alternative selling prices so we introduce new variables

x = revised price, - our decision variable,

Q = increase in sales per unit decrease in selling price,

E = extra sales resulting from a price change,

T = total sales after any given price change.

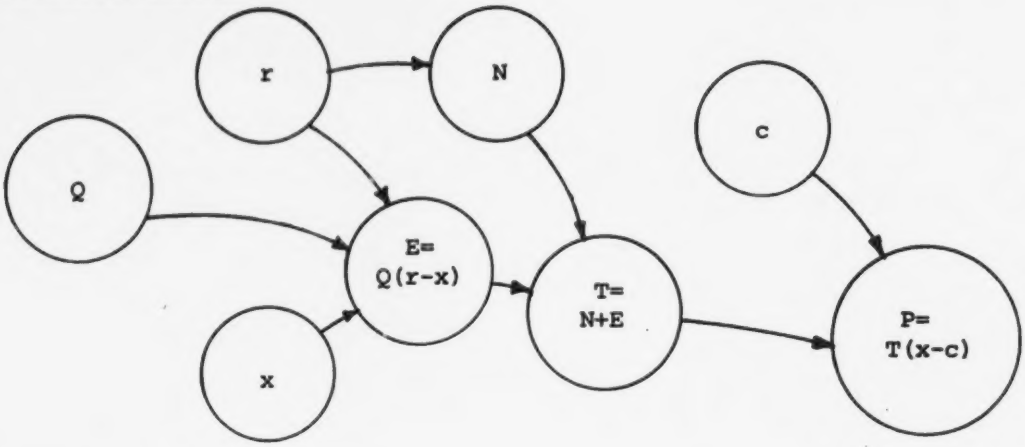
and we need two further assumptions:

4. The value Q is a constant term and is independent of N or x.

5. Q operates equally in an inverse manner. i.e. Increasing r by 1 unit results in a loss of sales of Q.

These final two assumptions are certainly open to question in that a small decrease in price may not result in many extra sales, while a larger change, perhaps below a significant value like £5.00 may have a disproportionate effect. Nevertheless an assumption that Q is constant is a good starting point.

3. INFLUENCE DIAGRAM



selling price = x,

decrease in price = r - x,

extra sales yielded, E = Q(r - x),

net profit per sale = x - c,

total sales after price change, T = N + Q(r - x),

the total net profit expected

$$P = \left[N + Q(r - x)\right](x - c)$$

4. MODEL DEVELOPMENT

Clearly some reduction in price will yield higher sales and possibly higher profits but there must be a limit.

A large price reduction, whilst increasing sales, diminishes profitability. Indeed we can get larger sales by selling at below cost price but there would be a negative profit.

To maximise profit, we consider P as a function of x.

i.e. Let $P = f(x)$

$$= \left[N + Q\,(r - x)\right](x - c)$$

where N, Q, r and c are constants.

Expanding gives

$$P = Nx - Nc + Qrx - Qx^2 - cQr + cQx.$$

Rearranging this quadratic function of x,

$$P = -Qx^2 + \left[N + Q(r + c)\right].\ x - c(N + Qr).$$

To optimize, we differentiate and equate the differential to zero,

$$f'(x) = -2Qx + N + Q(r + c)$$

$$2Qx = N + Q(r + c)$$

The optimum selling price is

$$\boxed{X_o = \frac{N}{2Q} + \frac{(r + c)}{2}}$$

Whilst the optimum profit is

$$\boxed{P_o = \left[N + Q(r - X_o)\right](X_o - c}$$

For pupils unfamiliar with calculus, the optimization could be done graphically or algebraically by completing the square.

Analysis

The model gives an optimum price,

$$X_o = \frac{N}{2Q} + \frac{r + c}{2}$$

the variables r and c are fixed by external factors.

N is a factor of the environment. All three are easily determined.

The value of Q however, is not so readily found. It is therefore reasonable to ask what part a possibly indeterminate value has upon the model.

Many products, including essential or specialist goods are not very vulnerable to price change, and would therefore, have relatively small values of Q. Such examples include petrol on motorway service stations, certain strongly patented business machines, curios in gift shops and many luxury items. For items such as these

if Q is very small $\frac{N}{2Q}$ is large

i.e. $X_o = \frac{r + c}{2}$ + some large value

This implies that with Q sufficiently small, the optimum price may well be greater than the recommended price, indeed X_o may well be very large. Such a strategy is not unknown and is usually referred to as 'profiteering'.

Gift shops attached to places of interest and ice-cream or hamburger stalls are examples of the way in which this price mechanism operates.

Increasing the price naturally loses a proportionate number of sales but the profits are increased by the higher prices paid by the remaining customers.

Sensitivity analysis

We have looked at the relationship between the inputs to the model and seen their effect on the variables.

Now we must consider the accuracy of the inputs and the effects of any errors that they may contain. As has already been stated elsewhere, N, r and c are easily determined quite accurately, X_o is a product of the optimum price model. The one questionable input variable is Q.

Q is a very real variable in any marketing situation but whilst we may guess at its value we can never be certain of our accuracy.

Indeed we can NEVER know the real value of Q, nor can we be sure of our assumption that it is constant for a given set of conditions, in fact this assumption is quite probably incorrect. However, we can estimate the value of Q. How this might be done is outside the scope of this work although immediate thoughts tend in the direction of surveys and trials.

What is of concern here is the effect of any error in estimating Q. In other words, how reliable is our model? How sensitive is it to errors in estimating Q? What effect does an error in estimating Q have on optimal profit?

Let our estimate of Q be q

Then the estimated optimum selling price x and profit p is given by

$$x = \frac{N}{2q} + \frac{r + c}{2}$$

$$p = \left[N + q(r - x)\right](x - c)$$

$$= \frac{N^2}{4q} + \tfrac{1}{2} N (r - c) + \frac{q}{4}(r - c)^2$$

The actual profit P for the optimum selling price X is

$$P = \frac{N^2}{4Q} + \tfrac{1}{2} N(r - c) + \frac{Q}{4}(r - c)^2$$

Hence we can calculate how an error in estimating Q affects the error in estimating the profit either by considering P - p or $\frac{P}{p}$: the former seems slightly more tractable and meaningful

$$P - p = \frac{N^2}{4Qq}(q - Q) + \frac{Q - q}{4}(r - c)^2$$

$$= \frac{Q - q}{4}\left[(r - c)^2 - \frac{N^2}{qQ}\right]$$

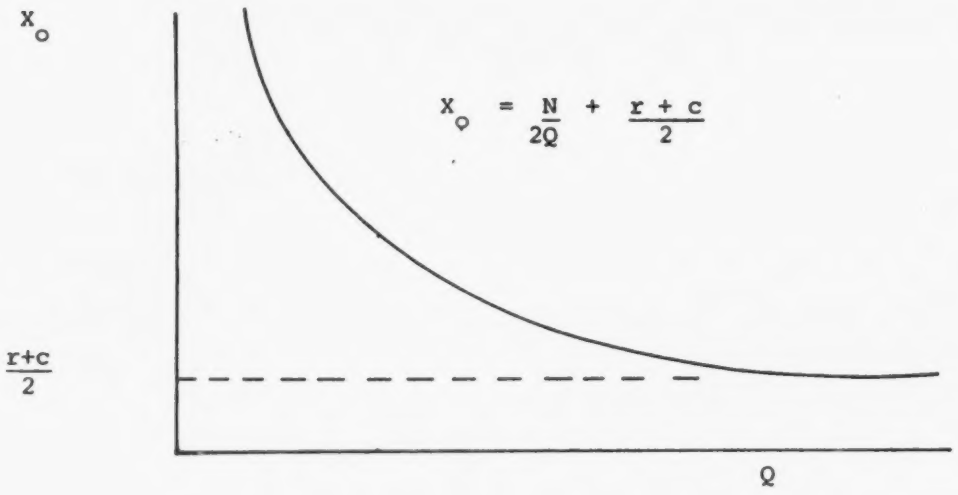
One could sketch this for various choices of parameters but this is unlikely to yield much more information.

Optimum price as a function

Looking again at the optimum price model, if we consider N, r and c as constants and Q as a variable we have X_o as a function of Q which is hyperbolic.

i.e. $$X_o = f(Q)$$
$$= \frac{N}{2Q} + \frac{r + c}{2}$$

and whose supremum and infimum are infinity and $\frac{r + c}{2}$.



It may be noted that if $X_o = r$,

$$r = \frac{N}{2Q} + \frac{r + c}{2}$$

$$Q = N/(r - c)$$

This means that when Q is equal to the current sales divided by the recommended profit, the optimum selling price is equal to the recommended price.

Self-assessment questions

As the model is aimed at average pupils, many of the questions below are straightforward numerical exercises to reinforce the ideas developed in the model. Clearly it is best to present each question just after the particular idea has been discussed (a typical solution is given for question 5).

1. A fruit cake currently sells (no pun intended) at 4000 per week, at a recommended price of 40p each. The cost price, inclusive of marketing, is 26p. A small survey indicates that for every 1p reduction in price, 500 more cakes will be sold.

   Calculate i) the current profit on sales. (£560)

   ii) the profit achieved by reducing the price of the cake by 2p and 10p respectively (£600, £360)

2. Find the optimum selling price for the fruit cakes mentioned above and the expected profit (37p, £605)

3. Find the optimum selling price and profit and the percentage increase in profit by optimisation, for the fruit cake example, given the value for Q of 100,000 (33p, £49,280, 8700%).

4. Show that by reducing the price to 32p, (more than half the recommended profit) a loss of profit is to be expected (£1040).

5. Find the optimum selling price and profit for the fruit cake example given a value for Q of 50 (73p, £1104).

   Solution

   $N = 4000, \quad Q = 50, \quad r = 40, \quad c = 26.$

   Optimum selling price

   $$\underline{x} = \frac{4000}{2(50)} + \frac{40 + 26}{2}$$

   $$= 40 + 33$$

   $$= 73p$$

   Optimum profit

   $$\underline{P} = \left[4000 + 50(40 - 73)\right] (73 - 26)$$

   $$= (4000 - 1650) \times 47$$

   $$= 2350 \times 47p$$

   $$= £1104.50$$

N.B. Raising the price to 73p naturally loses many customers (1650), but still increases the profit (£560) by a maximum amount of £544.50.

6. Consider the optimum selling price and profit when Q = 1. (This would imply that the product sales are very insensitive to price change). (£20.33, £40 280).

7. A chain of department stores currently sell 10,000 jackets a month throughout the country. Each jacket costs £15 and is selling for £20. From a sample of stores an estimate figure for Q of 4000 is obtained with an upper limit of 20% error.

   Calculate: a) the current profit (£50 000)

   b) the optimum profit if Q is correct (£56.250).

   c) the optimum profit if Q is 20% higher (£60,208).

5. CONCLUSION

This model uses fairly simple techniques to develop an appreciation of a realistic problem. The main assumption about the linearity of the main parameter Q can lead to an interesting discussion.

One could go on to assume a more complicated function for Q, perhaps with upper and lower bounds. There is also the possibility of collecting data, perhaps just before and during a sale period to indicate how well the model approaches the real situation.

V POSITIONING A REAR-SCREEN WIPER

Geoff Sheath

1. INTRODUCTION.

Anyone who drives will have an opportunity to study the raw material of this problem all too frequently. The rear-screens of hatch-back cars now often have windscreen wipers and you may have wondered at one time how the size and position of the single rear-screen wiper is decided. To the casual observer some cars seem to have so inadequate an arrangement that the wiper must, one feels, have been added without any real regard to its effectiveness, whilst other cars have complex looking bent arms and off-centre pivot points that look on the contrary as if considerable thought has gone into their design.

Models for some of these arrangements and their characteristics are analysed in terms of some general parameters. Foundations are also laid for a more specific analysis that could be carried out on an individual car rear-screen.

The problem is interesting because, although it is fairly describable using 2D geometry, there are far too many possible variables to proceed generally without cutting down on them. Sketches are used to examine different possibilities and decide between them. One aim of the problem is thus to show students how informal decision processes based on likelihood and common sense can eliminate possible but unlikely configurations.

Even when the variables have been reduced to manageable proportions an analytic approach to maximization and sensitivity is difficult and fairly unenlightening. The problem cries out for a numerical/graphical treatment which, using even a straightforward programmable calculator, is fairly easy. So another aim of the model would be to develop students' feeling for numerical solutions and their graphical representation.

Age-Range : 15 - 18 years

Pre-requisites: Simple trigonometry

2. INITIAL ASSUMPTIONS.

1. The screen is flat and the pivot of the arm is perpendicular to it.

   This enables the model to consist of entirely 2D geomtry. In practice the rear-screen is fairly flat on most cars.

2. The wiper can be set to sweep right up to the edges of the screen if necessary.

   In practice manufacturers seem to leave a small gap unswept (to avoid the blade hitting the edge and perhaps wearing out rapidly). This can be encompassed within the model by taking this border off measurements.

3. There is no restriction on manufacturing any particular type of arm.

4. The rear-screen is rectangular.

   Most rear-screens are more closely approximated by trapezia than by rectangles. However, by selecting a value for the width of the screen between the maximum and minimum widths the rectangle model may be used quite effectively.

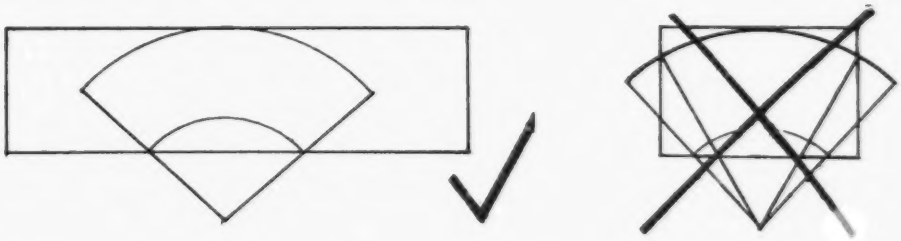
5. The best position for a rear-screen wiper is that which sweeps the largest area.

   This model takes no account of whether it is more valuable to sweep one part of the screen than another.

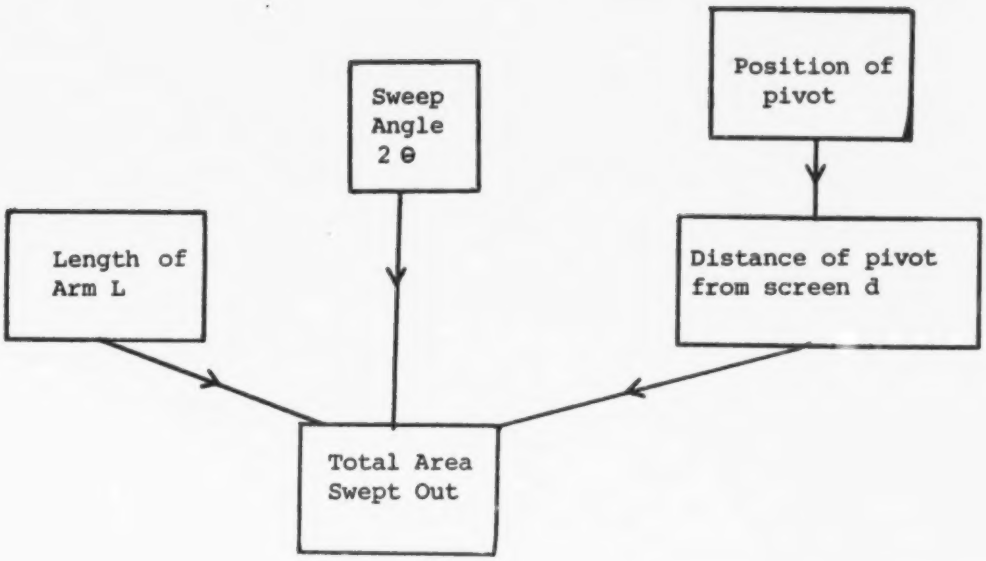
6. The wiper blade runs along the wiper arm.

   The blade is not angled with respect to the arm.

7. The screen is wide enough for the sweep of the blade to be unrestricted by the sides of the screen.

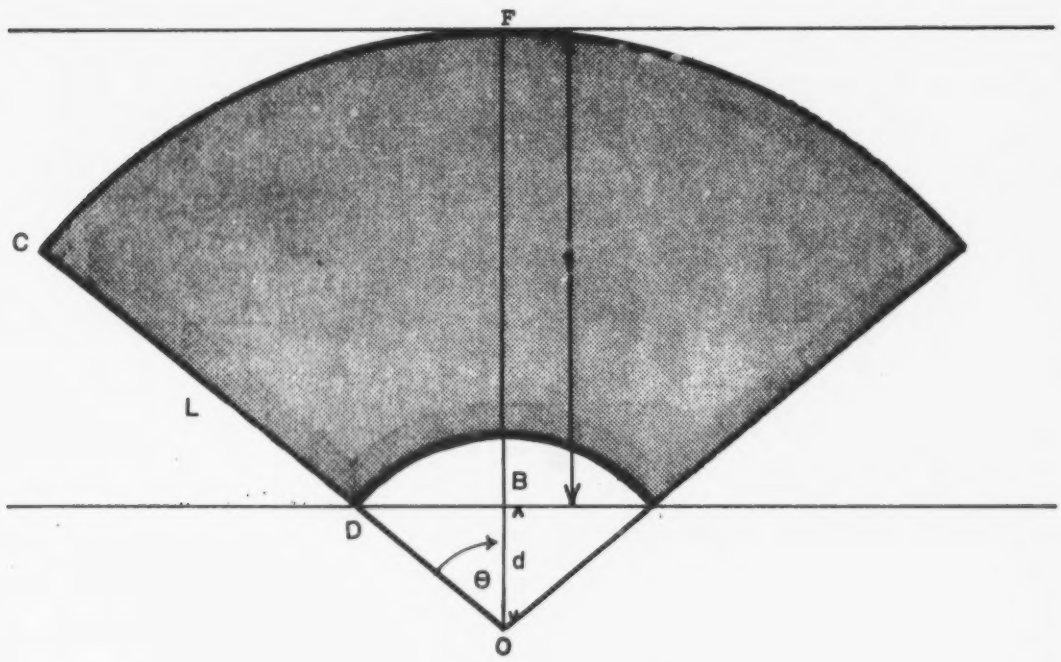


3. **INFLUENCE DIAGRAM.**



4. MODEL DEVELOPMENT (1).

For the greatest area the blade should just touch the top of the screen.



The depth of the screen may be taken as 1 and other measurements can then be expressed in terms of it. This reduces the number of variables by one.

d = distance of pivot from edge of screen: OB

θ = half the angle of sweep: DOB

L = length of the wiper blade: CD

A = area swept by the wiper blade

Length of arm = FO = 1 + d
= CO = L + d.sec θ

$$\theta = \cos^{-1}\frac{BO}{OD} = \cos^{-1}\frac{d}{1 + d - L} \qquad (1)$$

$$A = \frac{2\theta}{360} \cdot (\pi\, OC^2 - \pi\, OD^2)$$

$$= \frac{\theta\pi}{180}\left[(1 + d)^2 - (1 + d - L)^2\right] \qquad (2)$$

Equation (1) gives θ as a function of d and L while equation (2) gives A as a function of d,L and θ. Hence for any given d and L, A can be found.

The question now is: 'What d and L maximise A ?' '. For $L \to 1$ and $d \to \infty$, $A \to \infty$ so there is no strict maximum. But, of course, in practice d cannot tend to ∞: the values of d measured from the photographs (shown later) range between 0.09 and 0.35. So a sensible approach would seem to be to take particular values of d and, for each, find the value of L which maximises A.

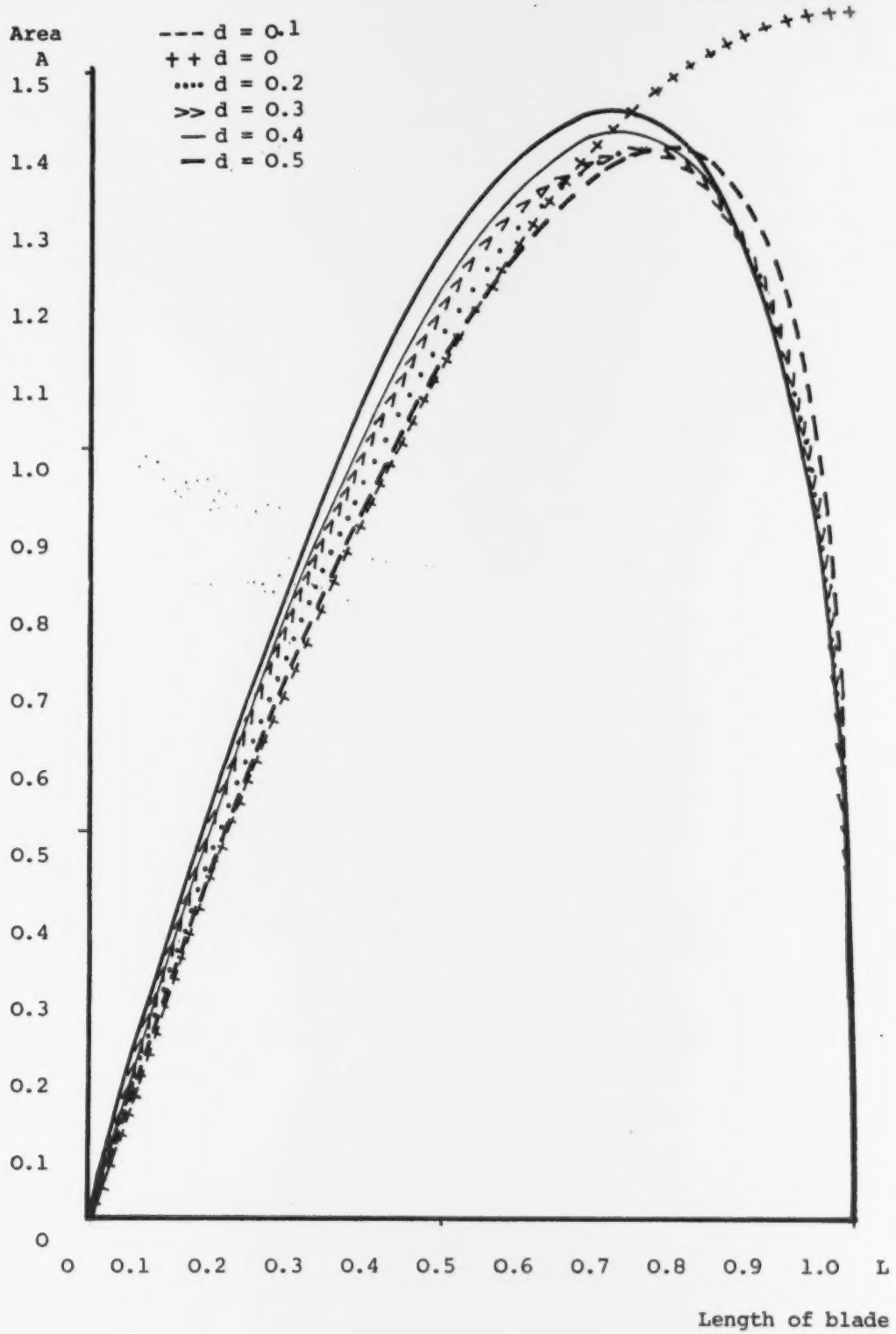
This could be done analytically by substituting θ from (1) into (2) and differentiating. But the algebra gets very complicated. Just finding maxima would be difficult, to do any sensitivity analysis well nigh impossible. Approximating the function by a similar one may be possible but this does not seem a promising line to pursue, partly because there are no obviously simple candidates, partly because errors would anyway have been difficult to deal with.

Instead a numerical solution was attempted using a Commodore PR100 programmable calculator. This calculator can be programmed to perform a sequence of up to 72 key strokes automatically, using data stored in any of 10 memories. (By using the program the value of d is placed in memory 2, the value of L in memory 1 and the programme will then calculate A.)

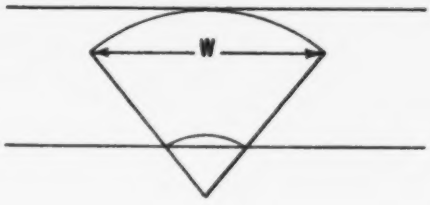
For d = 0, 0.1, 0.2, 0.3, 0.4 and 0.5 values of L were taken from 0 to 1 in 0.1 steps at first, and then at finer intervals as necessary to draw the graph overleaf accurately.

Fig 1

Graph to show variation of Area swept with length of blade for various values of d.



Here is a table to show, for a given value of d, the maximum area swept A_{max} and, to achieve that the area, the length of blade (L_{Amax}), angle (θ_{Amax}) and width of area swept, where



w is the actual width of the area swept

$w = (1 + d) \sin \theta$. This gives the minimum width required to sweep the maximum area.

d	A_{max}	L_{Amax}	θ_{Amax}	W
0	1.57	1	90°	2
0.1	1.39	0.75	73°	2.1
0.2	1.38	0.73	65°	2.17
0.3	1.39	0.71	59°	2.24
0.4	1.42	0.68	56°	2.33
0.5	1.44	0.68	52°	2.38

These values can be obtained by reading from the graph to give L_{Amax} and then drawing an accurate diagram (or using trigonometry) to find corresponding values for the angle and the width.

Sensitivity Analysis

From the table and the graph it can be seen that, except for d = 0, the area swept reaches a maximum at about L = 0.7 with a maximum area of about 1.4. With the special case d = 0 a full 180° ($\theta = 90^{\circ}$) sweep is possible and therefore the maximum area is swept when the blade goes from the pivot to the top of the screen.

When $d > 0$, a full sweep is not possible as the blade must stay on the screen. The optimum length of blade is about 0.7 (remember it takes the depth of screen as 1) giving a maximum area swept of about 1.4. As d increases, the optimum length of blade and angle swept decreases, while the width and area swept increase.

In fact the maximum area swept is not very sensitive to changes in d. If d increases from 0.2 to 0.4 (by 100%), the corresponding increase in A_{max} is only 0.04 (about 3%). Similarly, for a fixed value of d, changes of L from L_{Amax} lead to comparatively small changes in A. With d = 0.5, for example, $A \geq 95\% A_{max}$ for $0.55 \leq L \leq 0.8$ i.e. for $L \pm 17\%$. Also, if $0.1 \leq d \leq 0.5$ and $L = L_{Amax} \pm 0.1$ then $A \geq 90\% A_{max}$.

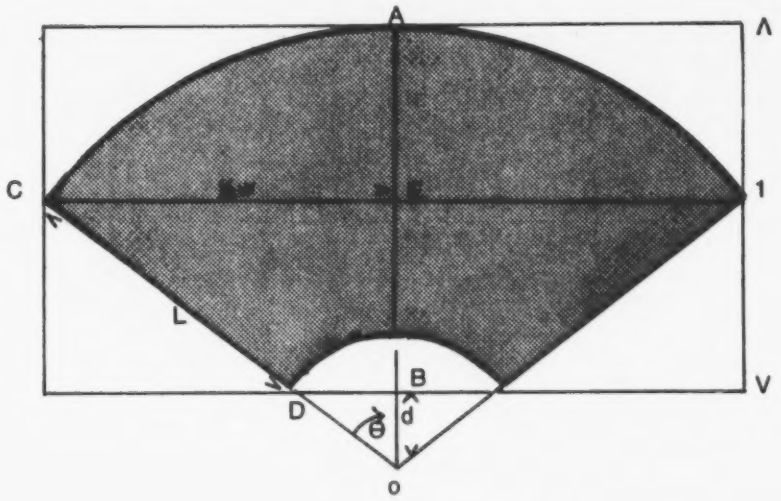
These figures can be derived by reading from the graph or by numerical calculation. It is particularly worthwhile for pupils to undertake this sensitivity analysis as it shows the role played by each of the variables. It illustrates how sensitive (or insensitive) a particular variable is, an important part of the modelling process.

5. MODEL DEVELOPMENT (2)

In the model above, we assumed that the screen is wide enough for the sweep of the blade to be unrestricted by the sides of the screen. We now drop this assumption 7. For this model, we assume 1-6 as above and

8. The maximum area is swept out when the pivot is placed centrally.

(In fact this assumption could be proved.)



From the diagram we obtain

$$\text{Sin } \theta = \frac{CE}{OC} = \frac{\frac{1}{2}W}{L+d\sec\theta} = \frac{\frac{1}{2}W}{1+d} \qquad (3)$$

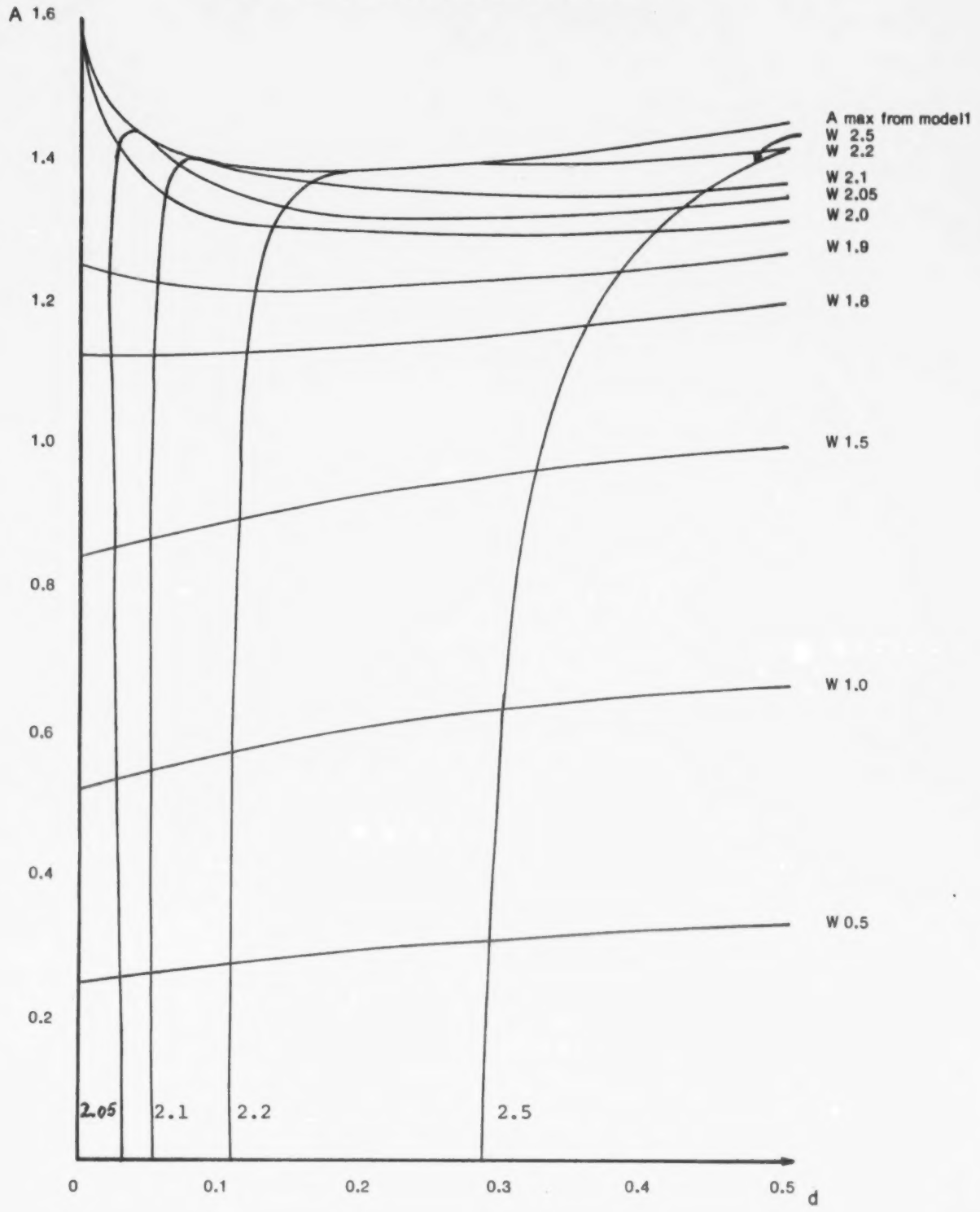
Since OD = $d\sec\theta$, we find using equation (2),

$$A = \frac{\pi\theta}{180}\left[(1+d)^2 - (d\sec\theta)^2\right] \qquad (4)$$

Remember that the main difference from model 1 is that the width of the screen is assumed to restrict the angle of sweep which is possible. These two equations enable the area swept to be calculated from d and w. This was done by the calculator program and the results are plotted on the graphs overleaf.

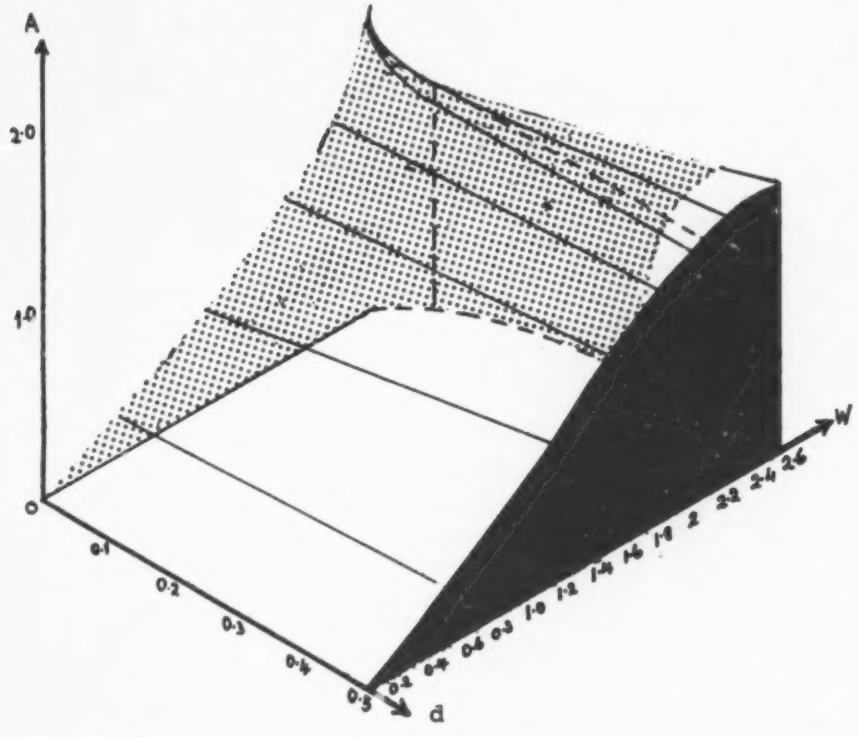
Fig 2

Graph to show how A varies with d, given w



The same results can be plotted in three dimensions, as shown below:-

Fig 3



Sensitivity Analysis

From these two graphs, there are three different types of values for w which require consideration. (screen area = w)

(a) $w \leq 1.8$

A increases with d, so d should be chosen as large as possible. Note that, although A increases by approximately the same amount (0.1) between d = 0 and 0.5 (w =0 .5 to 1.5) the proportion of screen swept is more sensitive to d for small w (e.g.; w =0..5 d = 0, 44% swept; d = 0.5, 66% swept) than for large w (e.g.; w = 1.5, d = 0, 55% swept; d = 0.5, 67% swept).

(b) $1.8 < w \leq 2$

There is a minimum A between d = 0 and 0.5
For w = 1.9, A is not very sensitive to changes in d.
(3% difference between A_{max} amd A_{min} over the range $0 \leq d \leq 0.5$)

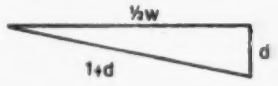
For w = 2, for $d < 0.1$ choose d as close to 0 as possible
for $0.1 \leq d \leq 0.5$ A is insensitive to changes in d.

(c) $w > 2$

The wiper cannot touch the side of the screen unless

$(1 + d)^2 \geq (\frac{1}{2}w)^2 + d^2$

i.e. $d \geq \frac{1}{8}w^2 - \frac{1}{2}$



As d increases from this value the graph of A climbs steeply until it touches the curve of A_{max} from model 1. The graph then either dips to a minimum and then increases slowly, or just increases slowly. So d should be chosen near the value giving the maximum but A is not very sensitive to increases in the value of d.

The first model relates θ and A to d and L by the equations

$$\theta = \cos^{-1} \frac{d}{1 + d - L} \qquad (1)$$

$$A = \frac{\theta \pi}{180} \left((1 + d)^2 - (1 + d - L)^2 \right) \qquad (2)$$

In these equations the width of the screen is ignored. It is assumed to be wide enough for a full sweep of the blade.

The second set of equations deals with the case where the width may be a restriction, and hence gives A and θ in terms of d and w.

$$\theta = \sin^{-1} \frac{\frac{1}{2}w}{1+d} \qquad (3)$$

$$A = \frac{\pi \theta}{180} (1+d)^2 - (d\sec\theta)^2 \qquad (4)$$

For wide screens with $w > 2$ we cannot get values of θ from equation 3 for some d as $\sin\theta > 1$. We therefore need to use equation 1 and 2 to calculate θ and A.

If w and d are such that equation 3 gives a value of $\sin\theta < 1$ then equation 3 and 4 can be used to calculate θ and A.

The next section uses these two approaches in some examples taken from actual cars.

To see how reasonable the models so far produced are, we can measure actual cars and see how their dimensions compare with the prediction of the models. The lengths are all relative as the depth of screen in each case is taken to be 1.

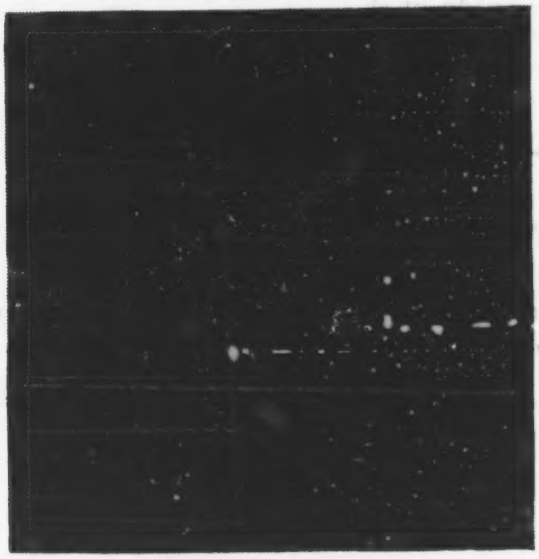
VALIDATION

Example 1: Volvo

Measurements: d = 0.11, w = 2.5. Width is not a restriction.
We can use equations 1 and 2 for Θ and A

These predict maximum area swept of 1.4 when blade length L = 0.76

The measured L = 0.82 gives A = 1.38.



Example 2: Marina

Measurements: d = 0.35, w = 3.24. Width is not a restriction.
Model 1 gives a maximum area swept of 1.41 when blade length is 0.68
The measured L = 0.76 gives A = 1.38

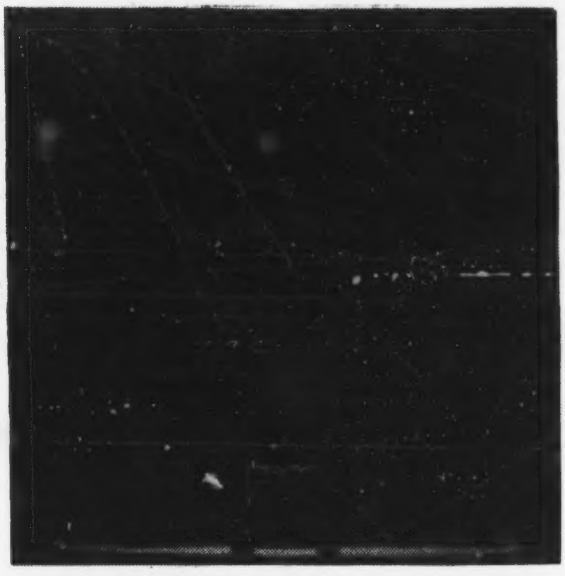
The area swept could have been increased by using a longer arm
The maximum area would have then occured when L = 0.7 and would have been 1.39 (50% of screen) with respect to the new unit depth.

Example 3: Ford

Measurements: d = 0.27, w = 2.8. Width is not a restriction.

We must use equations 1 and 2 for Θ and A

These predict maximum area swept of 1.39 with L = 0.7
The measured L = 0.87 gives A = 1.22
Noticeable improvement of area swept could be obtained by shortening the wiper blade. But the pivot would have to be centralised.

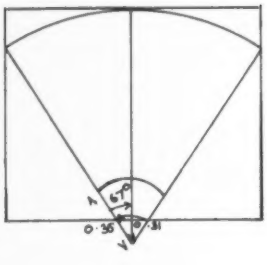


Example 4: MG

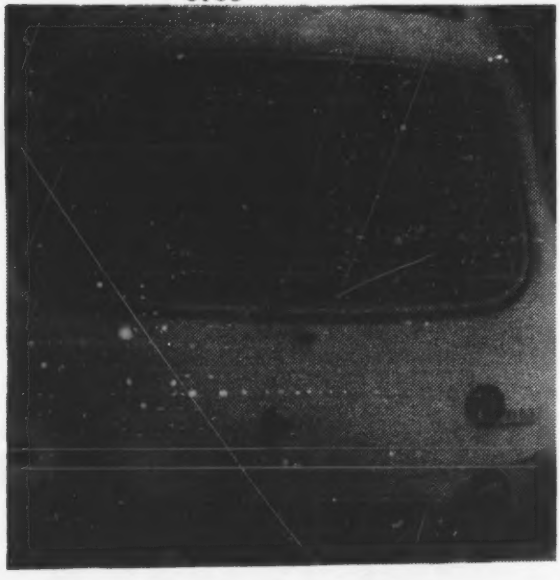
Measurements: d = 0.12, w = 2.06. Width is a restriction

We must use equations 3 and 4 for Θ and A

These give A = 1.36, L = 0.81, $\Theta = 67^{o}$
The measured L is 0.77, but little area is lost



$$\text{Area lost} = \frac{67}{180}\pi \left[(1.12 - 0.77)^2 - (1.12 - 0.81)^2\right]$$
$$= 0.03$$



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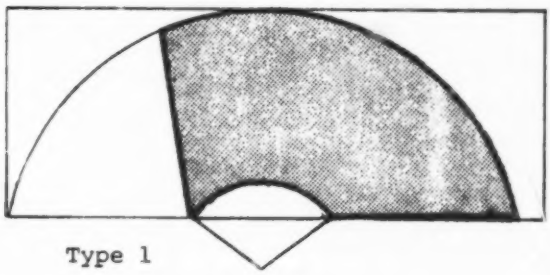
6. MODEL DEVELOPMENT (3)

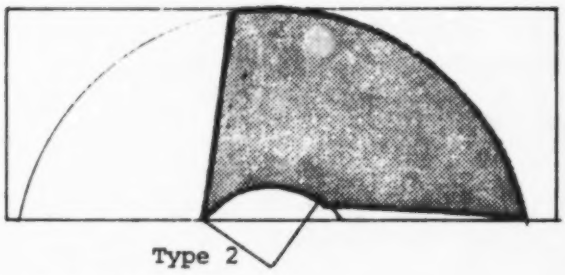
In our development so far it has been assumed that the blade runs along the wiper arm (assumption 6).

If this assumption is relaxed then there are a variety of possible configurations which can result:

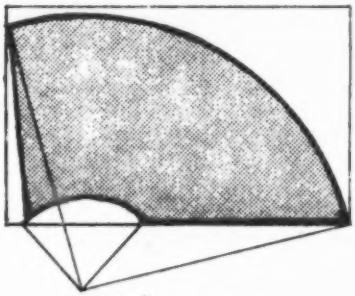
Some configurations possible if the wiper blade is angled to the wiper arm



Type 1



Type 2



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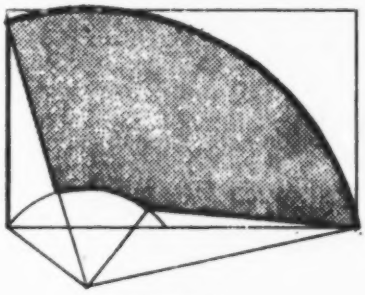

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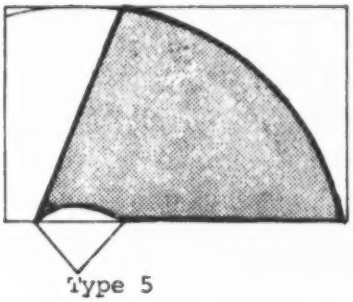
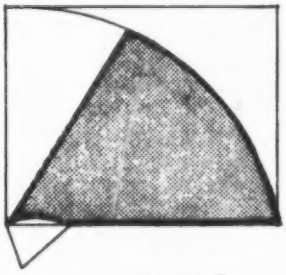


Type 5



Type 6

Each type of configuration could be modelled but this would be a lengthy process. The diagrams therefore give clues as to which to examine and these choices are expressed formally as assumptions.

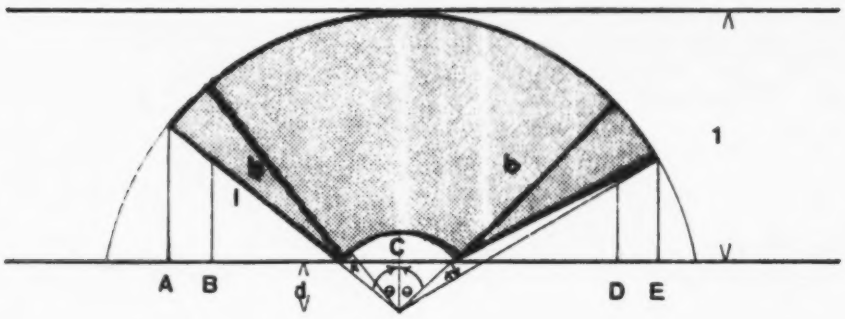
A model based on diagram 1 is developed below.

Assumptions

1 - 5 and 7 from model 1

8. The angle K the blade makes with the arm is less than or equal to 90-θ.

This assumption rules out configurations of type 2.

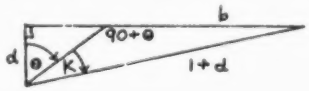


For any given d and L the area swept out is the same as for model 1 since the area not swept on the left (horizontal shading) exactly equals the additional area swept out on the right (vertical shading) so (except for the width) the maximal conditions still apply.

However, the width of the screen needed in model 2 is
$BE = BC + CE = (l + d)\ (\sin(\theta - k) + \sin(\theta + k)) = 2\ (l + d,\ \sin\theta.\cos k$
which decreases as k increases.

So the minimum width occurs when k is maximum, which is when the blade lies along CE, when

$$\cos(\theta + k) = \frac{d}{l+d}$$



and, by the sine rule, the length of blade

$$b = (l + d).\ \frac{\sin k}{\cos\theta}$$

and the distance of the pivot (horizontal) from the screen edge is

$(l + d)\ \sin\ (\theta + k)$

In model 1, when d = 0.5 the maximum area (1.44) occurs when

$L = 0.68$, $\theta = 52^{\circ}$ and $w \geq 2.38$.

For model 2 when d = 0.5 the maximum area is still 1.44 and

$L = 0.68$ and $\theta = 52^{\circ}$

However

$$\cos(52 + k) = \frac{0.5}{1.5}$$

$k = 19^{\circ}$

so $w_{min} = 2 \times 1.5 \times \sin 52 \times \cos 19$
$= 2.24$

and the blade length

$$b = \frac{1.5 \times \sin 19}{\cos 52}$$

$= 0.79$

Thus, by inclining the blade 19° to the arm it is possible to sweep an area of 1.44 on widths of screen down to 2.24.

Here are other results derived from the results for model 1:

d	A	L	θ	MODEL 1 W_{min}	MODEL 3 W_{min}	b	Distance of pivot from edge	k
0	1.57	1	90°	2	2	1	1	0°
0.1	1.39	0.75	73°	2.1	2.06	0.77	1.1	12°
0.2	1.38	0.73	65°	2.17	2.10	0.73	1.18	15°
0.3	1.39	0.71	59°	2.24	2.12	0.78	1.27	18°
0.4	1.42	0.68	56°	2.33	2.22	0.77	1.34	18°
0.5	1.44	0.68	52°	2.38	2.24	0.79	1.42	19°

7. CONCLUSION

Three distinct models have been developed.

The first two models apply when the wiper blade lies along the wiper arm.

In these cases we can derive equations for the area which are functions either of the blade length or of the screen width.

The solutions given by the model were compared with examples taken from photographs of cars.

The results were:

Example 1 - VOLVO - centrally located arm, large area swept, in fairly good agreement with the model.

Example 2 - MARINA - arm off centre, large area unswept, model shows that a larger arm would give a better result.

Example 3 - FORD - arm off centre, sweep constrained by side. Considerably less area than possible swept out - a shorter arm centrally located would give better result.

Example 4 - MG - arm in centre, narrow screen. Close to maximum area possible.

An investigation was then made of the possibilities of the wiper blade being inclined at an angle to the wiper arm, and this indicates improvements in area that can be swept out.

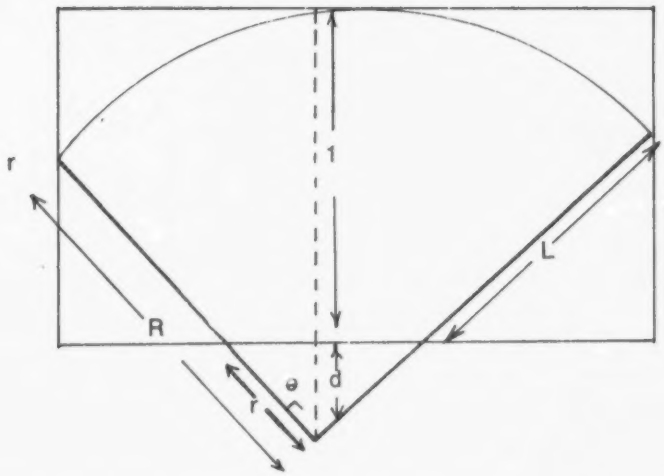
These models provide a useful exercise in the numerical solution of a problem which is too intractable for analytical solution.

One reason why this has occurred may be the choice of variables. The two lengths that were used were L and d in the first model, and it may be that other lengths may give more tractable relationships.

The choice of the 'right' variables for a model can often be crucial, if the aim is to get an equation which is easily maximised. In this case the variables chosen lead to an equation which was best dealt with by numerical models.

Readers are invited to try out other possibilities to see if they can obtain other equations for A. One possibility is to try $R = l + d$ and $r = l + d - L$, which give

$$A = \frac{\theta \pi}{180}(R^2 - r^2) \text{ and } \cos\theta = \frac{R - l}{r}$$



The results given by the model depend heavily on the validity of the assumptions.

Assumption 1 is reasonably close to the truth in practice. Assumptions 2, 3 and 4 are reasonable.

Assumption 5 is interesting, especially when we look at the examples photographed, as in two of them the wiper is clearly off-centre, and obviously has been designed to maximise the area through which the driver looks.

The Ford succeeds much better than the Marina in this respect. A further model could be done by varying assumption 5, and, regarding one half of the screen as more important, to seek to maximise covering this half.

Other configurations

There are other wiper systems used on lorries and buses that might make the basis for a self-assessment project. An easy example is the parallel linkage used on this bus.

V1 THE LONG JUMP

Mike Sheridan

1. INTRODUCTION

In track and field Athletics the Running Long Jump provides us with one of many real life situations which can be modelled mathematically.

In the long jump a competitor takes off on or behind a board known as the 'scratch line' and must travel as far as he can landing on sand. The length of the jump is the distance from the scratch line to the rearmost impression left by the jumper's body in the sand. Thus it is important for the jumper not to fall backwards.

In order for a competitor to be as successful as possible, what factors must he take into consideration in order to fulfil this aim and how significant are they ?

Traditionally the problem has been replaced by a mathematical treatise on projectile theory with the underlying implication that the mathematical formulation was self evident and that the method of solution would be justified by the fact that it had been well established.

However, this esoteric approach can be mystifying on first encounter by a student and, consequently, although he may be skilled in solving equations he may have little skill in formulating the model that they represent or, indeed, of interpreting their solutions.

The popular approach to teaching applied mathematics in schools until recently has been to treat the subjects in isolation to real-life applications. Hence it is the 18 year-old 'A' level student (or equivalent standard) to whom this particular example of an alternative approach to applied mathematics is aimed.

The following pages present three models of increasing degrees of complexity. Whilst it is true to say that the extremely crude Model 1 is an integral part of the development and refinement of a tractable solution to the problem, the material that it contains would be suitable for able 16 year-old 'O' level candidates who had a working knowledge of the quadratic and derived functions.

Models 2 and 3 on the other hand are likely to be suited to the afore-mentioned 18 year-olds, with the formulation and interpretation of these models aimed at the more able students. Indeed, the mathematics of Model 3 would rarely be handled confidently by students capable of less than 'A' or 'B' grade in the G.C.E.

Pre-requisites

1) 'O' level mathematics, including differentiation and integration of variables with integer indices and some applications (e.g. maxima and minima)
2) The solution to simple 1st order differential equations using the method of separation of variables
3) Resolution of vector quantities in two dimensional space.

Age-range 16-18 years.

2. INITIAL ASSUMPTIONS

The long jumper, at take-off, drives off from the board with either leg (he usually has a preference for which leg is best in contact with the ground). A novice jumper attempts to 'reach' as far forward as possible whereas an experienced jumper abandons this attempt to run-up and jump for distance. Instead he strives for the maximum height that his approach speed and musculature will allow at take-off. Why is this considered to be the favoured strategy over the one employed by the novice ? (here we have an optimisation problem). Hence we make the following assumptions.

1) Let us assume that the motion of the athlete from the instant of take-off to the moment when he lands is directed so that his centre of gravity, which can be considered as the point through which his effective weight passes, moves along his saggital plane i.e. his motion is two dimensional, hence no crosswinds affect his motion.

2) The resistance of the air is negligible and hence the 'flight path' is parabolic.

3) The jumper does not touch the ground beyond the scratch line in the act of jumping and lands without falling back.

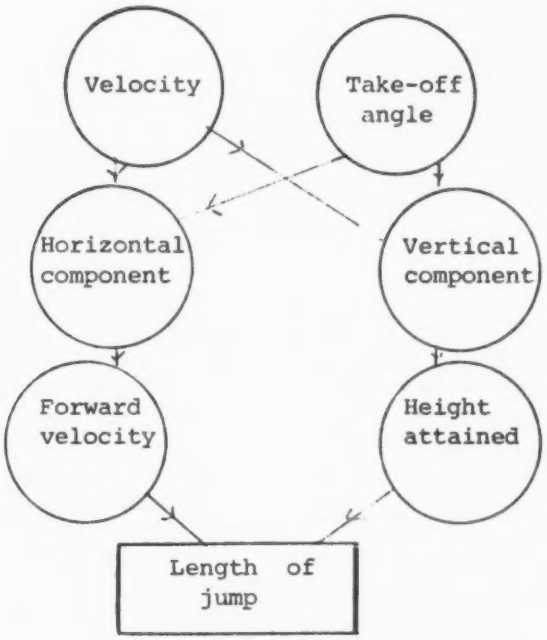
4) The competitor jumps exactly on the scratch line so that the distance measured is from take-off to landing.

5) The take-off board and jumping pit surface lie in the same horizontal plane.

6) Assume that the centre of gravity is directly over the scratch line at the instant of take-off and directly over the rear-most mark at landing and that, at these instants, the two positions of the C.G. lie in the same horizontal plane.

7) The maximum height reached is the only significant factor.

3. INFLUENCE DIAGRAM



4. MODEL DEVELOPMENT (1)

Let the scratch line of the take-off area be denoted by T and the rear-most part of the landing mark be denoted by L. The corresponding positions of the long jumper's centre of gravity are shown as T_1 and L_1 respectively in figure 1.

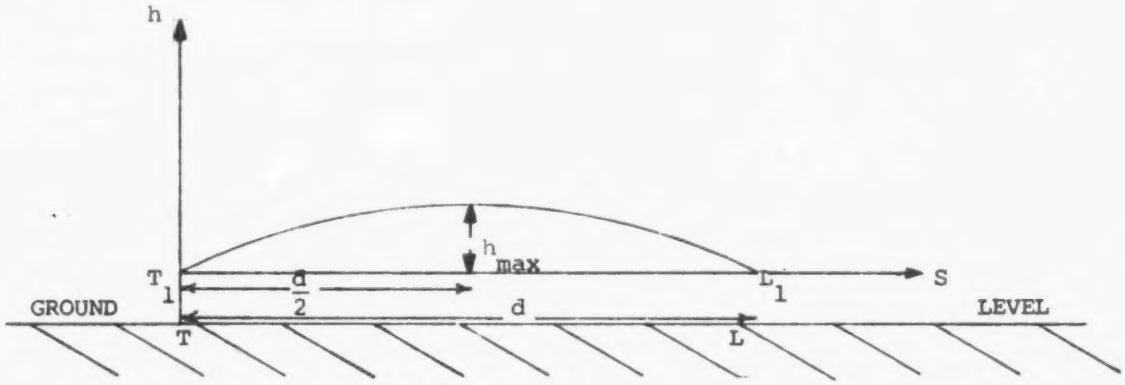


Fig. 1

From the general form of the graph above, (i.e. maximum at d/2, symmetrical, equal to zero at s = 0 and s = d), with a knowledge of the slopes of quadratic functions, the following function for h seems reasonable:

$$h = Ks(d - s) \qquad (1)$$

At the highest point of the jump $\frac{dh}{ds} = 0$

Thus $$s = \frac{d}{2} \qquad (2)$$

Combining (1) and (2) we obtain

$$h_{max} = \frac{k\, d^2}{4} \qquad (3)$$

Where h_{max} is the maximum height the jumper raises his centre of gravity after take-off.

Substituting K for $\frac{k}{4}$ in (3) we have

$$h_{max} = K\, d^2 \qquad (4)$$

Since K is a constant, the maximum height attained, during the course of the long jump, is directly proportional to the square of the distance travelled. In other words, all else being equal, from this model it would appear that common sense prevails in that the experienced jumper's attempts to get as much lift as he can at take-off is well founded.

Sensitivity Analysis

From cinematographic studies of certain first-class long jumpers (i.e. those who jump 7.90m or more) it has been found that they raise their C.G.'s no more than 75.90cm on their best jumps (information obtained from R.I.Brightwell, Lecturer in Physical Education, Loughborough College of Education, 1971).

Consequently if, for example, an athlete who jumped 7.90m raised his C.G. 75cm and, all else being equal, could only raise his C.G. 60cm on a subsequent jump then from (4) we would have

$$\frac{75}{(7.90)^2} = \frac{60}{d_2^2}$$

$$d_2 = 7.07\text{m}$$

where d_2 is the distance of this second jump. This would mean that his first jump of 7.9m would be reduced by approximately 80cm to a 'mere' 7.07m.

Thus this simple model illustrates the importance of striving for greater height in a long jump.

Comments

The athlete's coach may even desire to place an obstacle a certain distance from the scratch line to encourage the athlete to gain the desired height for a jump of 7.90m, say, If he regards a safe distance for the obstacle to be 2m, for example, (it may be regarded as safe because it would not be too near as to be off-putting and not so far that the likelihood of misjudgement is increased) for a jump of this distance (i.e. 7.90m) then using equations (1), (2) and (4), the height of the obstacle would have to be 56.7cm. (this ignores the feet).

Unfortunately, this model has a major weakness in that it stresses the need to strive for height only, in order to gain long distances, INDEPENDENT of any other factor. Whilst it is true that experienced long jumpers attempt to obtain height at take-off, it would be ludicrous to see them amble in at a leisurely pace and, with one massive effort, leap as high as they possibly could. Apart from the risk of damage to their health the nature of the event would have changed from a long jump to a high jump!

The model as it stands then does not entirely reflect the real-life situation. Indeed, more information is required, which in turn, leads to a further development in our model.

5. MODEL DEVELOPMENT (2)

In the real world we actually witness that the ability of the long jumper to spring in the air is coupled with a factor which is self-evident - his ability to run fast! With few exceptions, a good long jumper not only represents his club/team in this event but also participates in sprint races.

The question arises: "How is this velocity at take-off best applied in order to produce the best possible jump ?"

We thus alter assumption 7 and introduce assumptions 8, 9.

7) Since the motion occurs close to the surface of the earth we consider earth's acceleration due to gravity to be uniform.

8) The velocity of the athlete at the instant of take-off is independent of the corresponding tangential angle.

9) Newton's 'Laws of Motion' are correct.

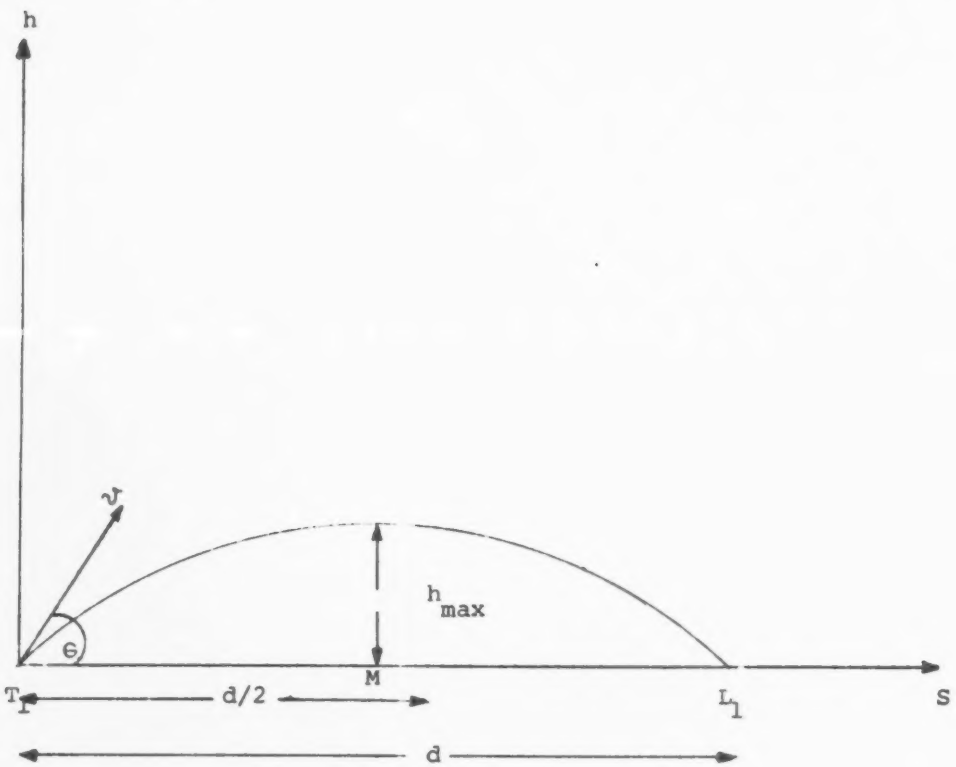


Fig. 2

Let the intial velocity of the jumper at take-off be v m/sec, making an angle of θ with the horizontal (see Fig. 2).

Using assumptions 2) and 9) the only force acting on the jumper is his weight. Consequently he moves vertically with uniform acceleration, $-g$ m/s^2.

Given that the time from the instant of take-off to the instant of landing is T seconds, then in the sagittal plane, after time t seconds, $0 \leq t \leq T$, we have the vertical acceleration

$$\frac{d^2h}{dt^2} = -g \qquad (1)$$

Upon integrating both sides of (1) with respect to t, we apply the initial condition that the vertical component of the velocity is given by $v\sin\theta$ m/sec. to obtain an expression for the vertical component of velocity after t seconds i.e.

$$\frac{dh}{dt} = v\sin\theta - gt \qquad (2)$$

Repeating the process, given that $h = 0$ when $t = 0$, then

$$h = vt\sin\theta - \tfrac{1}{2}gt^2 \qquad (3)$$

Alternatively, if $\frac{dh}{dt}$ is denoted by h' then

$$\frac{d^2h}{dt^2} = \frac{dh'}{dt} = \frac{dh}{dt} \cdot \frac{dh'}{dh} = h'\frac{dh'}{dh} \qquad (4)$$

Hence (1) and (4) go together to produce

$$h'\frac{dh'}{dh} = -g \qquad (5)$$

Separating the variables and integrating both sides, the D.E. becomes

$$(h')^2 = v^2\sin^2\theta - 2gh \qquad (6)$$

having used the initial conditions $h' = v\sin\theta$ when $h = 0$ to obtain the constant of integration $\frac{v^2\sin^2\theta}{2}$

In the horizontal plane

$$\frac{d^2s}{dt^2} = 0 \qquad (7)$$

$$\frac{ds}{dt} = v\cos\theta \qquad (8)$$

and $$s = vt\cos\theta \qquad (9)$$

having integrated both sides of the D.E.'s (7) and (8) respectively and having employed the initial conditions to give equations (8) and (9).

Consequently at L_1, $h = 0$, and, using (3), we obtain

$$T = \frac{2v\sin\theta}{g} \qquad (10)$$

From (9) and (10) the distance jumped in metres is given by

$$d = \frac{v^2\sin 2\theta}{g} \qquad (11)$$

Differentiating h in (3) with respect to t and setting it equal to zero, the maximum height, h_{max} is found when the time at maximum height,

$$t_{max} = \frac{v\sin\theta}{g} \qquad (12)$$

Substituting the value for t_{max} into (9) gives the distance in the horizontal direction when the maximum height is reached. Thus this distance

$$s_{max} = \tfrac{1}{2} v^2 \frac{\sin 2\theta}{g} \qquad (13)$$

Comparing (13) with (11) we can deduce that

$$s_{max} = \frac{d}{2} \qquad (14)$$

At the mid-point M,

$$t_{max} = T/2 \quad \text{(see Fig. 3)}$$

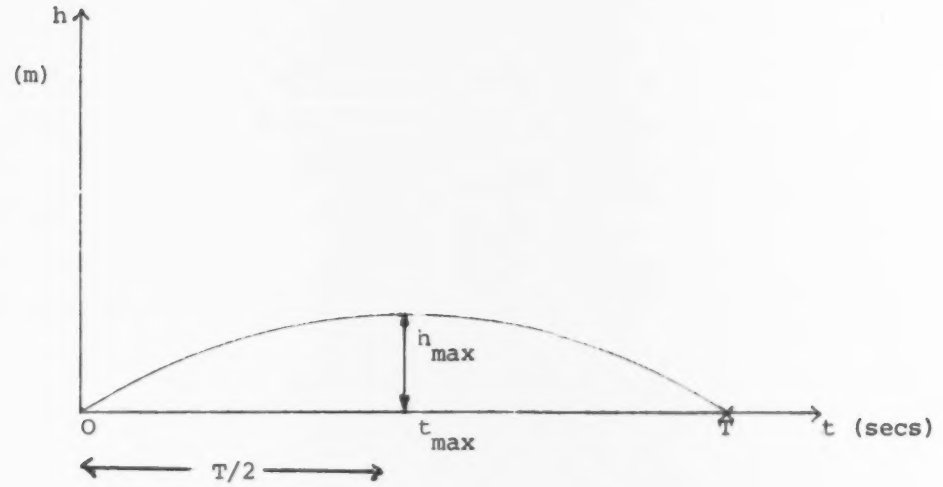


Fig. 3

When $t = t_{max}$, $\frac{dh}{dt} = 0$. Thus (6) provides us with the value

$$h_{max} = \frac{v^2 \sin^2\theta}{2g} \qquad (15)$$

(15) could also have been obtained by combining (12) with (13).

Using (9), (3) and (14) we find that the relationship between h_{max} and d is given by

$$h_{max} = \frac{\tan\theta}{2} d - \frac{g\sec^2\theta}{8v^2} d^2 \qquad (16)$$

with (15) giving the upper bound for h_{max}. Hence $d = \frac{v^2 \sin 2\theta}{g}$

Note also from equation (3) $h = vt\sin\theta - \frac{1}{2}gt^2$
and equation (9) $s = vt\cos\theta$
we obtain $h = s\tan\theta - \frac{g\sec^2\theta}{2v^2} s^2 \qquad (17)$

Remember, that for the first model we had

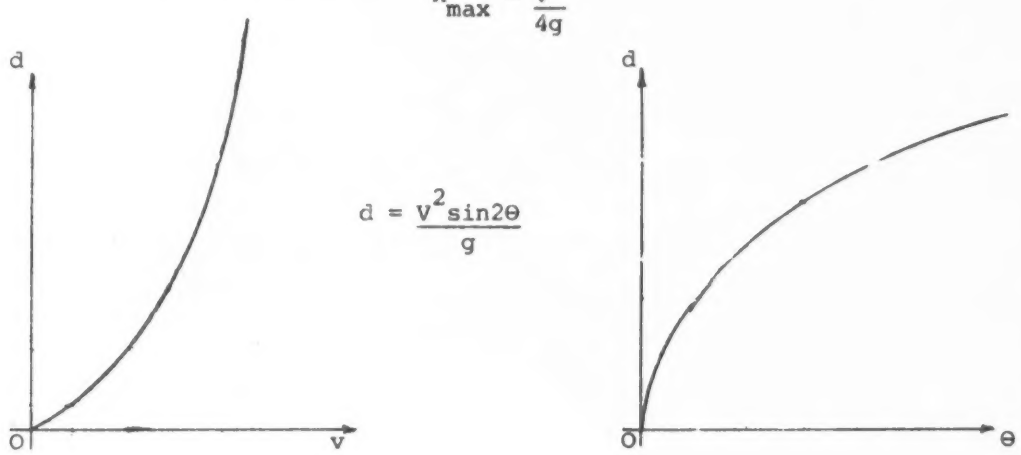
$$h = Ks\,(d-s)$$

By writing $K = \frac{g\sec^2\theta}{2v^2}$, we can express equation (17) in the form assumed in model 1 and this lends credibility to that model.

Analysis

We can now see that the distance travelled is more accurately determined by considering the two variables V and θ instead of the single variable h in model 1. From (11) which is illustrated in Fig 4, it is clear that the limit on the distance jumped occurs when $\theta = 45^{o}$ and that longer jumps are achieved by running faster (as one would expect intuitively). The maximum height possible when $\theta = 45^{o}$ is

$$h_{max} = \frac{v^2}{4g}$$



(a) (b)

Fig. 4

From figure 4 we can see the relationship between the distance jumped, d and

(a) the initial velocity, v, with the take-off angle θ held constant.

(b) the take-off angle, θ with the take-off velocity, v held constant.

On inspection these two relationships show that the initial take-off speed is much more significant than the corresponding tangential angle because the distance travelled is far more sensitive to changes in speed than to changes in the corresponding angles. For example, doubling the speed quadruples the distance whereas doubling the angle less than doubles the distance jumped.

Thus a long jump coach may well be advised to concentrate on improving the athlete's sprinting speed (assuming there is still room for improvement) before specializing on the angle at take-off.

Once this sprinting has been improved, by studying cinematographically the jumps of the athlete the coach may wait to improve the height reached by his jumper given that his approach speed is consistent from jump to jump. Knowing the best distances jumped, the horizontal velocity (which is constant throughout the flight) and the maximum height achieved on these jumps, he may wish to place an obstacle, such as a long piece of elastic, a safe distance from the take-off scratch line at a prescribed height. This would allow the jumper to achieve distances close to, or better than, his own personal bests. This training would then accustom the athlete to consistently jump long distances given that fatigue, illness, injury etc. do not occur.

For example, a safe distance for the obstacle may again be considered to be 2m for a 7.9m long jumper. For these jumps he might be capable of raising his C.G. 75cm whilst his horizontal velocity during flight is recorded as 7.9m/s (N.B. the best ever top sprinting speed is close to 10.97m/s i.e. approximately 39.5Km/h).

$$T = \frac{7.9}{7.9} = 1$$

Therefore the time of flight is 1 second

Now s and t are related linearly so that, when the athlete passes over the 2m mark, approximately ¼ of a second has elapsed. Consequently (3) leaves us with

$$h = \frac{v\sin\theta}{4} - \tfrac{1}{2} \cdot 9.8 \cdot \frac{1}{16} = \frac{v\sin\theta}{4} - .30625 \qquad (18)$$

taking g as $9.8 m/s^2$.
Since we are assumimg that the height raised in the jump is 75cm, equation (15) yields

$$.75 = \frac{v^2\sin^2\theta}{19.6} \text{ i.e. } v\sin\theta = \sqrt{14.7} \qquad (19)$$

Thus (18) and (19) give

$$h = .65$$

From this data the coach might like to consider placing the elastic 2m from the scratch line at a height of approximately 65cm.

Comment

This model again has its limitations because it assumes the C.G. of the jumper rises and falls equally, whereas in real life the C.G. of the jumper is approximately 30cm lower at the instant of landing than at take-off.

6. MODEL DEVELOPMENT (3)

We are now led to consider another factor. The position of the feet at landing was ignored in the previous two models. However, it would seem intuitively obvious that they should be held as high as is physically possible at landing so that they are well in advance of the centre of gravity of the jumper.

Let us first examine a refinement of model 2. From anthropometric data gathered in the U.S.A. on U.S.A.F. personnel it has been found that the average location for the C.G. of a male is approximately $\frac{11}{20}$ of his standing height. Again from cinematographic studies of the landing, the jumper's C.G. is approximately 30cm lower at the instant of landing than at the instant of take-off.

The assumptions are 1 to 5 in model 1 and 7, 8 and 9 in model 2 together with:

10. The jumper's heels are an extension of the flight path of his C.G. at the instant of landing. Thus we may consider the C.G. to end at ground level.

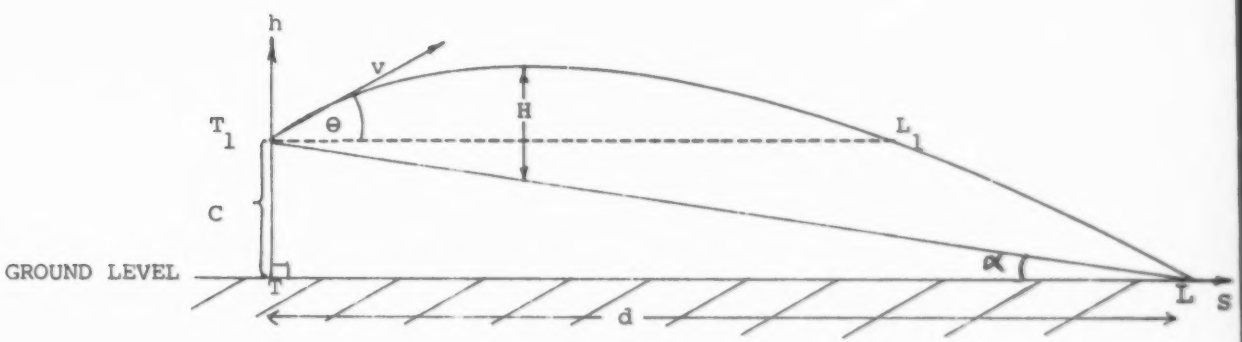


Fig. 5

Let the height of the jumper's C.G. above the ground at the instant of take-off be c m.

Upon resolving perpendicularly to the line T_1L (see figure 5) which is inclined at an angle α to the horizontal we obtain the following system of equations in the same fashion as for model 2

$$\frac{d^2H}{dt^2} = -g\cos\alpha \qquad (1)$$

$$\frac{dH}{dt} = v\sin(\theta+\alpha) - gt\cos\alpha \qquad (2)$$

$$H = vt\sin(\theta+\alpha) - \tfrac{1}{2}gt^2\cos\alpha \qquad (3)$$

At L, H = O. Using this value for H in (3) produces a new value for the time of flight given by

$$\frac{2\ v \sin(\theta + \alpha)}{g \cos \alpha} \qquad (4)$$

Resolving horizontally we have

$$\frac{d^2s}{dt^2} = 0 \qquad (5)$$

$$\frac{ds}{dt} = v \cos \theta \qquad (6)$$

$$s = vt \cos \theta \qquad (7)$$

as in model 2.

Equations (4) and (7) may be combined to give the total distance jumped measured along the horizontal line T L.

Hence

$$d = \frac{v^2}{g\cos\alpha} (\sin(2\theta + \alpha) + \sin \alpha) \qquad (8)$$

Note how this equation reduces to the earlier expression for d ($\frac{v^2 \sin 2\theta}{g}$) as $\alpha \to 0$, since $\sin\alpha \to 0$, $\cos\alpha \to 1$, $\sin(2\theta + \alpha) \to \sin 2\theta$.

This checking of the limiting case is stressed by Polya as an important step in problem solving.

Analysis

Using equation (13) from model 2 and the above equation (8) it is clear that the maximum height does not occur halfway through the jump as postulated in model 2. It is in fact, short of half way.

Let us consider a man whose height is 185cm since statistics from the 1964 Olympic Games show that most long jumpers are tall. From the U.S.A.F. findings his C.G. would be approximately 102cm when he stands erect. Now let us assume that at full extension at the instant of take-off this C.G. has been raised by 10cm (ignoring the small shift of the C.G. due to the change in moments of other parts of his body).

We can now take c as 112cm and consider our 7.9m long jumper once more. Let us assume that his take-off velocity is the same as before. Then from model 2 equation (19) and from the fact that his horizontal velocity is 7.9 ms^{-1}, V is found to be 78.2ms^{-1}. This seemingly sudden increase in velocity is due to the impulse applied at lift-off.

In $\Delta T_1 T L$ (figure 5)

$$\tan \alpha = \frac{1.12}{7.9} \qquad (9)$$

so $$\alpha = 8^{\circ}$$

Equations (8) and (9) enable us to find a value for θ,

$$\theta = 24^{\circ}\ 52' \qquad (10)$$

This value for θ enables us to calculate the maximum height reached by the C.G. during the course of the jump.

Model 2 equation (15) gives an expression for the maximum height, h_{max} attained with respect to the horizontal line $T_1 L_1$ i.e.

$$h_{max} = \frac{v^2 \sin^2\theta}{2g} \qquad (11)$$

Using our assumption 8 then if the jumper approaches the take-off board under the same conditions (same stride length, same leg cadence and consequently the same speed at take-off) then, from model 2 equations (9) and (19) combined with an assumed horizontal speed of 7.9ms^{-1}.

$$v = 78.2 \qquad (12)$$

Equations (10), (11) and (12) give us a different maximum height needed for the same jump of 7.9, and that is 70.5cm.

Using equation (12) from model 2 we find that

$$d_1 = \frac{v^2 \sin 2\theta}{g} \qquad (13)$$

where d_1 is the length of $T_1 L_1$.
Consequently the maximum height is located at a distance $\frac{d_1}{2}$ m

Thus, from equations (12) and (13) we find that the maximum height of the jump occurs approximately 3.03cm from the scratch line.

Now the time it takes to reach this distance is approximately 0.38 seconds (= t_1 secs, say), using equation (7) and, since the time and distance vectors are directly proportional, a safe distance of 2m would be passed at a time t_2 secs where

$$t_2 = \frac{2}{3.03} t_1 \quad \text{Hence} \quad t_2 = 0.248$$

Substituting this value for t_2 in equation (3) of model 2 we find that a coach placing the barrier 2m beyond the scratch line, would need to raise it to a height of 61cm for the athlete to produce consistent jumping of 7.9m.

Validation

This last model approaches reality much more closely than does its predecessors since it compares much more favourably with research findings from studies of slow-motion film and sequential 'stills' of long jumpers such as Igor Ter-Ovanesyan, U.S.S.R., Ralph Boston, U.S.A., Lynn Davies, G.B. and other consistent 7.9m jumpers in that it has been suggested that the proportion of horizontal to vertical velocity at take-off in good long jumping is very approximately 2:1. Their best ever sprinting times for 100 yards were 9.4, 9.6 and 9.5 seconds respectively which convert to average speeds of 9.8, 9.6, 9.7 m/s approximately. However, from these cinematographic studies it has been established that neither their take-off speed nor their spring are at a maximum owing to the limitations of the anatomical design of the human body. As stated before, they actually took off at less than 9m/s and raised their C.G.'s no more than 75-90cm. It is humanly impossible to jump at an angle of 45° without using a slow approach run or sacrificing horizontal speed by directing the take-off thrust backwards.

Up to this point model 2 could be said to be fulfilling the above criteria. The point for departure comes when one compares the two specifications for landing. Unless a jumper's technique deserts him it is highly unlikely that he will land with his legs vertically underneath him as he touches the sand. Long jumpers of high calibre are seen to extend their feet before them and, on landing, attempt to rotate their body over their feet in a crouched position. It is in this respect that model 3 is superior to model 2.

However, further refinements could be made to model 3 to improve its accuracy. For example, the distance of the C.G. of the athlete in front of the scratch line could be taken into account; a more accurate estimate for the true time of flight could be made if consideration was taken of the fact that the C.G. never actually touches the ground, since the take-off velocity depends upon its corresponding tangential angle, perhaps v could be expressed in some way as a function of θ, and finally, the effect of wind resistance (and assistance) could be incorporated into the model to make it even more true to life. Jumpers also move about in the air to improve their jump, but this is rather hard to model.

Self Assessment Questions

1. In the running long jump the athlete initiates forward rotation about his centre of gravity at the instant he loses contact with the ground. By "running in the air" he will absorb or displace rotation and produce an efficient landing position. Given an efficient landing is one in which the feet are placed ahead of the centre of gravity in such a way as to enable the jumper to rotate about the feet in a crouched position, until his centre of gravity comes to rest above this base, is it better to employ a one and one-half or a two and one-half stride hitch kick ? (A hitch-kick is the act of "running in the air")

2. In the running long jump the athlete is advised to try to keep his foot on the take-off board for as long as he possibly can. Is this a sensible thing to suggest ?

   Would the same advice be of benefit in the high jump ?

3. The rules of the high jump state that a competitor should attempt to clear a bar at a prescribed height without knocking it down from its supporting stands.

   The take-off involves a large vertical force which passes through the contestants centre of gravity.

   What is the best course of action to take in crossing the bar ?

7. CONCLUSIONS

In this work, the initial difficulty was to know exactly where the starting point should be in that 'throwing oneself in at the deep end' by immediately modelling the long jump in terms of parabolic motion might cloud one's vision as to how the model would be formulated together with the appreciation of the so-called underlying 'basic' assumptions. As Morris states:

> "Starting simply gets things moving..... It requires one to deliberately omit and distort certain aspects of the situation and to knowingly commit the sins of suppressing difficult situations and suboptimizing".

It was with this in mind that the very crude model 1 was introduced, with the minimum number of variables involved, so that one could concentrate more clearly on the underlying assumptions that might otherwise have been overlooked.

The simple quadratic function used to model the situation would be something that all 'A'-level candidates should be able to easily cope with, and indeed 'O'-level candidates should have little difficulty with the concepts involved as long as they are familiar with the algebraic properties of the parabola.

The next stage would be to discuss what the inputs for the model would be together with the corresponding outputs: in this case h_{max} is the dependent variable (or output) whilst s and d are our inputs.

Alas, owing to the over-simplification of the model with assumption 7) the credibility of the conclusion that the higher one jumps the further one will go without limitation is not without some doubt!

However, it can be said that the model reflects reality up to a point. One could then follow this statement up with a discussion upon where one would draw the dividing line for accepting the predictions provided by model 1.

The actual formulation of the problem, it should be pointed out, was deliberately suspect in that no limits or bounds were placed upon the variables under scrutiny. It was obvious, upon comparing the predictions with the real world, that the model needed further elaboration. Consequently there was a need to re-state the problem so that it would include some refinements which would more readily appeal to one's intuition. We would again need to explore the deductive tractability of the model and the assumptions which characterize it .

Model 2, then, was based upon a question which would encompass both the stated problem of model 1 and most of its assumptions together with three further assumptions. This model, like its predecessor, attempted to simulate the real situation and consequently lacked generality.

A typical problem-solving tool-analogy- was brought into play in as much as the motion of the jumper was taken to act through his C.G. and was directly related to the motion of a particle projected under gravity.

As the teacher, one could immediately call upon past experience and write down the relevant equations of motion but this would defeat the object of the exercise in the formulation and development of the model. The problem here was to assess how many new factors should be introduced. If there were too many then the model may have become far too unwiedly. The difficult decision lay in how precise we required the model to be.
The critical assumption that the C.G. rose and fell equally simplified the mathematics, yet data could be applied to the model and consistent results obtained from it. The one thing that it did lack was accuracy.

However, the important feature is that it did provide limits upon the increase in height which produce an increase in length as opposed to the limitless relationship between increase in height and increase in length found in the first model. This consequently fulfilled the expectation of the modeller.

The solving of equations, derived from the underlying assumptions, was streamlined for the express purpose of giving the pupils an opportunity to actively participate in the development of the problem at each stage and the sensitivity of the model was tested. From this some important implications were brought to light.

Firstly, by far the most significant factor found was the speed of the athlete. This led to an obvious implication for the coach. However, since competitions can be won or lost by a mere centimetre, there was strong evidence to suggest that increasing the lift at take-off, without sacrificing take-off velocity, was worth devoting some time to. This model also brought to mind as, indeed, did model 1, a method which might aid the attainment of this end - the planning of a (painless) obstacle at an appropriate distance and height.

Unfortunately, from practical considerations, this model was found to be unsatisfactory in that it did not tell the whole story. However, it did provide some insight into the problem. Consequently it was necessary to enrich model 2 by allowing for the C.G. being lower at the instant of landing than at the instant of take-off. This led to a dilemma. The assumptions that were introduced would not necessarily lead to a change in the system of equations that had been evolved throughout model 2. It would only mean an additional equation to take into account the height of the C.G. above ground level. The rest could be solved by the previous system if negative distances were used. Nevertheless, by resolving perpendicularly to the line joining the postulated end points of the C.G., a more compact expression for the time of flight could be obtained and likewise with an expression for the distance.

Later, in order to compare results, it was found useful to relate back to equations in model 2 and regard them as an integral part of model 3.

At this stage it may be wise to make a comparison of results from the three models.

Table 1 shows how a 20 per cent decrease in maximum height attained during the course of a 7.90m long-jump affects the distance jumped.

	MODEL 1		Decrease %	MODEL 2		Decrease %	MODEL 3		Decrease %
Distance Jumped	7.92m	7.09m	10	7.92m	5.73m	28	7.92m	7.57m	4
Maximum Height	75cm	60cm	20	75cm	60cm	20	70.49 cm	55.88 cm	20

TABLE 1

Although the 20 per cent decrease in height yields varying decreases in distances all three models are consistent in as much as both height and distance DECREASE. However, model 3 appears to be less sensitive to the decrease in height than do the preceeding models, but again it is important to remember that, as long as distances are measured to the nearest centimetre, any increase which is greater than or equal to that distance is significant and worthy of consideration. It is worth stressing at this point that this statement holds only if more significant factors, such as the take-off speed, have been taken to their physical limit.

After considering the merits of these models it would be foolish to suggest that this is the best possible reflection of the real world that we can obtain. Further questions should be asked such as:

"We have obtained an upper bound on θ. Can we now find a least upper bound".

The problem envisaged in trying to answer a question of this nature is that we may now be leaving the domain of the student's experience. A question of this nature implies searching for physical limits of human capabilities, which in turn implies a deep understanding of anatomy and physiology.

Other weaknesses are that changes in the surfaces to run-ups produce varying approach speeds and consequently affect each jumper in some way e.g. cinder versus rubber surfaces; also the model ignores psychological and motivational factors which account for fluctuations in performance.

However, as long as the modeller is aware of these limitations and puts them into perspective, the imaginary world that he produced can provide useful tools on which to base decisions, no matter how crude the model. After all, it is less expensive to base judgement on calculations which have been provided by a model than to use pure guesswork in situations where one has no experience.

VIII SUGGESTIONS FOR FURTHER READING

1. SOURCES OF MODELS

The practising teacher can find examples of models in books or journals like Bulletin of the IMA, and the International Journal of Mathematical Education in Science and Technology. In addition a specific journal has been established specially for this area : the Journal of Mathematical Modelling (JMMT) was founded in 1978 and, after four volumes, changed its name to Teaching Mathematics and its Applications. This journal provides a source of diverse interesting situations which could be modelled as well as discursive articles on educational issues. Many of the articles contain pertinent data which a teacher could use directly in the classroom.

While most of the models are designed for older students, with suitable adjustments they can often be simplified with lower objectives for younger or less experienced pupils. As a starting point, a selected list of models arising in different topic areas is given below. This is followed by a substantial list of references.

TOPIC AREA	REFERENCE
A. Biology	
Population	Burghes and Read (1978)
Drug administration	Penrose (1978)
	Saunders (1979)
Drug concentration	Burghes (1980)
Wrens dying in winter	Nelson (1977)
B. Management and Marketing	
Optimum order quantity	Burghes and Read (1978)
	Penrose (1978)
	Saunders (1979)
Depot location	Burghes (1980)
Retail price index	Schools Council Project (1981)
Advertising strategy	Burghes (1980)
Failure rate of printing colour	Nelson (1977)
Layout of items in store	Pollak (1968)
Telephone charges	James and Wilson (1983)
C. Sport	
Angle for the shot putt	Burghes and Read (1978)
Backspin on golf shots	Nelson (1977)
World records in Athletics	Burghes (1979)
Mathematics and Rowing	Townend (1982)
Pole-vaulting	Sheridan (1980)

D. Physics

Planetary motion	Burghes (1980)
Forces - Action - Reaction	d'Inverno and Mclone (1977)
U-tube accelerometer	Oke and Bajpai (1982)
Car braking systems	Clements (1980)
Record player Stylus *	Oke and Bajpai (1982)
Speed-wobble in Motorcycles	Oke in James and McDonald (1981)

* An example of a model developed for advanced students but adapted by O'Hare (1980) for use with C.S.E. pupils.

E. Transport

Seat belts	Burghes (1980)
Speed in tunnels	Pollak (1968)
Timing of traffic lights	Burkhardt (1978)
Parking	Pollak (1968)

F. Domestic

Moving an object round a corner	Pollak (1968)
Heating a baby's bottle	Oke (1979)
Collecting sets of cards	Pollak (1968)
Sizes of books	Pollak (1968)

G. Miscellaneous

Population and rank of cities	Nelson (1977)
Legal statements	Pollak (1968)
Gravity models in geography	Ling (1977)
Evacuation of a school	Wilson (1983)
Apportionment of seats in an election	Burghes (1983)

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